NOTES ON THE TOPOLOGY OF MAPPING CLASS GROUPS

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Abstract. This is a short collection of notes on the topology of big mapping class groups stemming from discussions the author had at the AIM workshop "Surfaces of infinite type". We show that big mapping class groups are neither locally compact nor compactly generated. We also show that all big mapping class groups are homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Finally, we give an infinite family of big mapping class groups that are CB generated and hence have a well-defined quasi-isometry class.

Before beginning, the author would like to make the disclaimer that the arguments contained in this note came out of discussions during a workshop at the American Institute of Mathematics and therefore the author does not claim sole credit, especially in the case of Proposition 10 in which the author has completely borrowed the proof.

For the entirety of the note, a *surface* is a connected, oriented, second countable, Hausdorff 2-manifold. The mapping class group of a surface S is denoted $\mathrm{MCG}(S)$. A mapping class group is *big* if the underlying surface is of infinite topological type, that is, if the fundamental group of the surface cannot be finitely generated.

1.1. **Topology of mapping class groups.** Let $\mathcal{C}(S)$ denote the set of isotopy classes of simple closed curves on S. Given a finite collection A of $\mathcal{C}(S)$, let

$$U_A = \{f \in \mathrm{MCG}(S) : f(a) = a \text{ for all } a \in A\}.$$

We define the *permutation topology* on $\mathrm{MCG}(S)$ to be the topology with basis consisting of sets of the form $f \cdot U_A$, where $A \subset \mathcal{C}(S)$ is finite and $f \in \mathrm{MCG}(S)$. This topology agrees with the quotient topology coming from $\mathrm{Homeo}^+(S)$ equipped with the compact-open topology (see Appendix A for a discussion of this viewpoint).

The first thing to note about the topology of big mapping class groups is the following (reformulation of a) result of Hernández, Morales, and Valdez.

Theorem 1 ([11, Corollary 1.2])**.** *Let S be a surface of infinite type and let $\mathcal{F}(S) = \{A \subset \mathcal{C}(S) : |A| < \infty\}$. Then*

$$\bigcap_{A \in \mathcal{F}(S)} U_A = \{id\}.$$

Proposition 2. *Every big mapping class group is separable and metrizable.*

Proof. Let S be an infinite-type surface. If, in a topological group, the identity can be separated from any other point by an open set, then this is enough to guarantee the group is Hausdorff; hence, $\mathrm{MCG}(S)$ is Hausdorff by Theorem 1. Moreover, by the countability of $\mathcal{C}(S)$, it is clear that the permutation topology is second countable and hence first countable and separable. The Birkhoff-Kakutani theorem says that every Hausdorff, first-countable topological group is metrizable. □

Note that another description of the permutation topology is as the coarsest topology for which the orbit map $\varphi_c\colon\ \mathrm{MCG}(S) \to \mathcal{C}(S)$ given by $\varphi_c(f) = f(c)$ is continuous for every $c \in \mathcal{C}(S)$, where we view $\mathcal{C}(S)$ as a discrete topological space. As an immediate consequence of this description, we have:

Proposition 3. *If S is of infinite type, then* $\mathrm{MCG}(S)$ *has a basis of clopen sets (i.e. it is a zero-dimensional space).* □

The *curve graph* of a surface S is the set $\mathcal{C}(S)$ together the edge relation defined by $a \sim b$ if a and b have disjoint representatives. We will also denote the curve graph by $\mathcal{C}(S)$. The *extended mapping class group* of a surface S, denoted $\mathrm{MCG}^{\pm}(S)$, is the group $\mathrm{Homeo}(S)/\textit{isotopy}$.

As in the finite-type setting, it has been shown by Hernández, Morales, and Valdez and also independently by Bavard, Dowdall, and Rafi that automorphisms of the curve graph are induced by mapping classes:

Theorem 4 ([12, 4])**.** *Let S be an infinite-type surface. The automorphism group of the curve graph $\mathcal{C}(S)$ is the extended mapping class group* $\mathrm{MCG}^{\pm}(S)$.

A topological space is *Polish* if it is separable and completely metrizable. A *Polish group* is a topological group whose underlying topological space is Polish.

Corollary 5. *If S is an infinite-type surface, then every closed subgroup of* $\mathrm{MCG}^{\pm}(S)$ *is Polish.*

Proof. In Proposition 2 we saw that $\mathrm{MCG}(S)$ is separable and metrizable, so we only need a complete metric. The same proof holds for $\mathrm{MCG}^{\pm}(S)$. Enumerate the elements of $\mathcal{C}(S)$ by the natural numbers and define $d\colon\ \mathrm{MCG}^{\pm}(S)\times\mathrm{MCG}^{\pm}(S) \to \mathbb{R}$ by

$$d(f,g) = \inf_{n\in\mathbb{N}\cup\{0\}} \{2^{-n}\colon\ f(c_k) = g(c_k) \text{ for all } k < n\}.$$

Now let $\rho\colon\ \mathrm{MCG}^{\pm}(S) \times \mathrm{MCG}(S) \to \mathbb{R}$ be the map given by $\rho(f,g) = d(f,g) + d(f^{-1},g^{-1})$. It is a standard exercise in descriptive set theory to prove that ρ is a complete metric (note: this is a general construction for automorphism groups of countable structures). It is then easy to see that every closed subspace of a separable, complete metric space is again a separable, complete metric space. □

The symmetric group on $\mathbb{N}$ letters can also be given the corresponding permutation topology and the same proof as the above corollary shows that it is a Polish group. In fact, the proof of the above corollary actually yields a stronger statement about mapping class groups:

Corollary 6. *The (extended) mapping class group of every surface is isomorphic (as topological groups) to a closed subgroup of the symmetric group on $\mathbb{N}$ letters.* □

The goal of the rest of the note is to investigate whether big mapping class groups have a well-defined metric up to quasi-isometry, or, in other words, to investigate whether the tools of geometric group theory apply to big mapping class groups. The first issue is that big mapping class groups are uncountable and hence not finitely generated. The next best case is to have a compactly-generated locally-compact group as the standard results of geometric theory are known to hold in this setting. Unfortunately, we now see that big mapping class groups are neither:

Proposition 7. *If S is of infinite type, then every compact subset of* $\mathrm{MCG}(S)$ *is nowhere dense.*

Proof. Suppose $V \subset \mathrm{MCG}(S)$ is compact with non-empty interior. By translating, we may assume that V contains the identity in its interior. Therefore, there exists a finite subset $A \subset \mathcal{C}(S)$ such that $U_A \subset V$. Choose $c \in \mathcal{C}(S)$ disjoint from each curve in A. The Dehn twist T_c is in U_A and the sequence $\{T_c^n\}_{n\in\mathbb{N}}$ contained in U_A is a discrete and closed subset of V, a contradiction. □

Corollary 8. *If S is of infinite type, then* $\mathrm{MCG}(S)$ *is neither locally compact nor compactly generated.*

Proof. It is immediate from Proposition 7 that $\mathrm{MCG}(S)$ is not locally compact. Now let V be a compact subset of $\mathrm{MCG}(S)$. Note that V^n is compact for every $n \in \mathbb{N}$ and hence nowhere dense by Proposition 7. In particular, $\bigcup_{n\in\mathbb{N}} V^n$ is a union of nowhere dense sets in a Polish space and hence has empty interior; thus, V cannot generate $\mathrm{MCG}(S)$. □

As an aside to the current discussion, we can use our work thus far to give a concrete description of the topology of $\mathrm{MCG}(S)$.

Corollary 9. *If S is of infinite type, then* $\mathrm{MCG}(S)$ *is homeomorphic to the Baire space* $\mathbb{N}^{\mathbb{N}}$ *(which in turn is homeomorphic to the space of irrational numbers* $\mathbb{R} \smallsetminus \mathbb{Q}$*).*

Proof. By the Alexandrov-Urysohn theorem (see [13, Theorem 7.7]), every non-empty Polish zero-dimensional space for which every compact set has empty interior is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$. □

We have already seen that $\mathrm{MCG}(S)$ is not compactly generated (Corollary 8), but we give here another – more hands on – proof that does not rely on knowing that $\mathrm{MCG}(S)$ is Polish.

Proposition 10. *If S is of infinite type, then* $\mathrm{MCG}(S)$ *is not compactly generated.*

Proof. (The proof below was presented during the AIM workshop by Spencer Dowdall, but there was a larger working group that came up with the argument.) Let $V \subset \mathrm{MCG}(S)$ be compact. Equip $\mathcal{C}(S)$ with the discrete topology. By definition of the permutation topology, $\mathrm{MCG}(S)$ acts on $\mathcal{C}(S)$ continuously. Therefore, the orbit map $\mathrm{MCG}(S) \to \mathcal{C}(S)$ is continuous; so,

$$V \cdot c = \{f(c) : f \in V, c \in \mathcal{C}(S)\}$$

is compact and hence finite for any $c \in \mathcal{C}(S)$. It follows that $V^n \cdot c$ is finite as well for every $n \in \mathbb{N}$.

Choose a sequence $\{(a_i, b_i)\}_{n\in\mathbb{N}}$ satisfying $a_i, b_i \in \mathcal{C}(S)$, $i(a_i, b_i) > 0$, and $i(a_i, a_j) = i(a_i, b_j) = i(b_i, b_j) = 0$ for all $i, j \in \mathbb{N}$ such that $i \neq j$. Now for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $T_{b_k}^{n_k}(a_k) \notin A^k(a_k)$, where T_{b_k} is the Dehn twist about b_k.

Now let

$$g = \prod_{k\in\mathbb{N}} T_{b_k}^{n_k}.$$

Suppose there exists $n \in \mathbb{N}$ such that $g \in V^n$. Observe that $g(a_k) = T_{b_k}^{n_k}(a_k)$ for all $k \in \mathbb{N}$; however, by construction, for $k > n$ we have $T_{b_k}^{n_k}(a_k) \notin A^k(a_k)$, a contradiction. □

1.2. **CB generated Polish groups.** Fortunately, the recent work of Rosendal provides a framework greatly enlarging the class of groups that have a well-defined quasi-isometry type. We give a very brief description in the special setting of Polish groups.

A subset A of a topological group G is *coarsely bounded*, or *CB*, if it has finite diameter in every left-invariant continuous pseudo-metric on G. In the setting of Polish groups, a useful characterization of coarsely bounded sets can be given:

Proposition 11 ([16, Proposition 2.7])**.** *Let G be a Polish group. For every identity neighborhood V in G, there exists a finite set $F \subset G$ and $k \in \mathbb{N}$ such that $A \subset (FV)^k$ if and only if A is coarsely bounded.*

Before continuing, we need some terminology.

Define an order on left-invariant pseudo-metrics on a topological group G by declaring $d_1 \ll d_2$ if $d_1 < K \cdot d_2 + C$ for some constants K and C. Note that any two metrics that are maximal with respect to this ordering are quasi-isometric. A metric on a topological space is *compatible* if it the metric topology agrees with the topology of the space.

A topological group is *CB generated* if it has a coarsley-bounded generating set. A set in a Polish space is *analytic* if it is the continuous image of another Polish space.

The following theorem puts together pieces of Theorem 1.2, Proposition 2.52, Theorem 2.53, and Example 2.54 from [16] to get a statement suited to our goal, which is to establish a well-defined quasi-isometry class of a CB-generated Polish group.

Theorem 12 (Rosendal)**.** *Let G be a a CB-generated Polish group. Let A be an analytic coarsely-bounded generating set of G.*

(a) There exists a left-invariant compatible maximal metric on G quasi-isometric to the word metric associated to A. (Hence, every maximal metric is quasi-isometric to the word metric associated to A.)

(b) If B is another analytic coarsely-bounded generating set of G, then the word metrics associated to A and B are quasi-isometric.

Note that the metric topology associated to a word metric is always discrete and hence cannot be compatible with a non-discrete topological group. However, the above theorem tells us that given a word metric (associated to a coarsely bounded generating set), we can find a left-invariant compatible metric in its quasi-isometry class. We can therefore talk about the quasi-isometry type of a coarsley-bounded Polish group.

1.3. **CB generated mapping class groups.** With the language of Rosendal laid out in the previous section, the question now becomes:

> *Which (if any) infinite-type surfaces have CB-generated mapping class groups?*

The above question was recently resolved by Mann-Rafi [14].

The goal of what follows is to exhibit a (countably) infinite family of surfaces whose mapping class groups are CB generated. The author believes it is accurate to attribute the origin of the ideas in the proofs of Theorem 13 and Lemma 14 to Kathryn Mann and Kasra Rafi.

Theorem 13. *If S is an infinite-genus surface with a finite number of ends, none of which are planar, then* $\mathrm{MCG}(S)$ *is CB generated.*

Before getting to the proof, we give a lemma that provides us with many coarsely-bounded subsets of $\mathrm{MCG}(S)$.

We call a finite-type connected subsurface whose boundary components are all separating, essential curves and whose complementary components are all unbounded a *star surface*. Given a star surface K in a surface S, we can view $\mathcal{C}(K)$ as a subset of $\mathcal{C}(S)$ and $\mathrm{MCG}(K)$ as a subgroup of $\mathrm{MCG}(S)$. For a star surface K, define

$$U_K = \{f \in \mathrm{MCG}(S) : f(a) = a \text{ for all } a \in \mathcal{C}(K)\}.$$

It is not hard to see that U_K is a clopen subset of $\mathrm{MCG}(S)$ (in fact, there exists a finite subset A of $\mathcal{C}(S)$ such that $U_K = U_A$).

The following proofs will rely heavily on a homeomorphism called a *handle shift* that was introduced in [15]. For the definition and details about handle shifts we refer the reader to either [15] or [1].

Lemma 14. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let S be an infinite-genus surface with n ends, all of which are non-planar. If K is a star surface in S with n boundary components, then U_K is coarsely bounded in* $\mathrm{MCG}(S)$.

Proof. Let $U = U_K$. We will use the characterization of coarsely bounded given in Proposition 11. Let V a neighborhood of the identity. By replacing V with a smaller neighborhood of the identity, we may assume, without loss of generality, that there exists a star surfce Σ with n boundary components such that $K \subset \Sigma$ and

$$V = U_\Sigma = \{f \in \mathrm{MCG}(S) : f(a) = a \text{ for all } a \in \mathcal{C}(\Sigma)\}.$$

Let $f \in U$ and let $X_1, \ldots, X_n$ denote the closures of the components of $\Sigma \setminus K$. For each $i \in \{1, \ldots, n\}$, there exists a handle shift h_i and an integer m_i such that

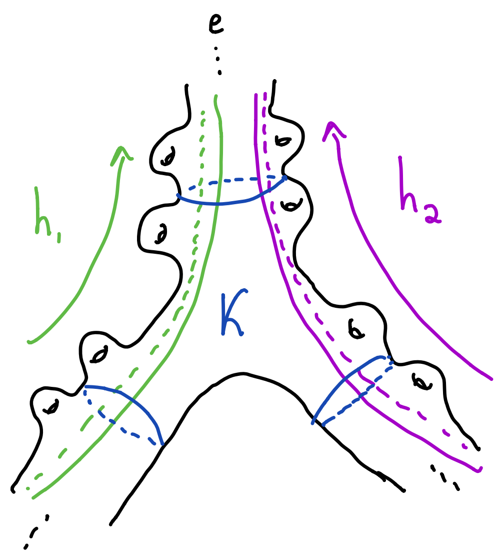

FIGURE 1. A 3-ended surface with star surface K and the handle shifts h_1 and h_2 represented from the proof of Theorem 13.

$h_i^{m_i}(X_i \cup f(X_i)) \cap \Sigma = \emptyset$. Now, $h_i^{m_i}(Y_i)$ is homeomorphic to Y_i, where Y_i is the component of $S \smallsetminus K$ containing X_i. Therefore, we can find a homeomorphism g_i supported in $h_i^{m_i}(Y_i)$ such that $g \circ h_i^{m_i} \circ f(a) = h_i^{m_i}(a)$ for every $a \in \mathcal{C}(X_i)$. Let $g_i = h_i^{-m_i} \circ g^{-1} \circ h_i^{m_i}$ and note $g_i \in \langle F \cup V \rangle$ as $g \in V$. Now $f(a) = g_i(a)$ for every $a \in \mathcal{C}(K \cup X_i)$.

Notice that the supports of the g_i are pairwise disjoint and hence we can define $g = g_1 \cdots g_n$ and it follows that $g(a) = f(a)$ for all $a \in \mathcal{C}(\Sigma)$ as $\Sigma = K \cup X_1 \cup \cdots \cup X_n$. In particular, $g^{-1} \circ f \in V$. Now let $k = 1 + n + 2\sum_{i=1}^{n} |m_i|$. We have shown that $U \subset (FV)^k$ and hence U is coarsely bounded. □

We now proceed to the proof of Theorem 13:

Proof of Theorem 13. If S is the Loch Ness Monster surface, that is, if it is one ended, then Mann and Rafi have proved that $\mathrm{MCG}(S)$ is coarsely bounded [14], so we will assume that S has at least two ends.

Let n denote the number of ends of S and choose a labelling of the ends $\{e_1, \ldots, e_n\}$. Choose a star surface K of S with genus $2(n-1)$ and with n boundary components. Choose $n-1$ pairwise-commuting handle shifts $h_1, \ldots, h_{n-1}$ satisfying the following conditions

(i) $h_i^+ = e_n$ for every $i \in \{1, \ldots, n-1\}$,

(ii) $h_i^- \neq h_j^-$ for $i \neq j \in \{1, \ldots, n-1\}$,

(iii) $S \smallsetminus \bigcup_{i=1}^{n-1} \mathrm{supp}(h_i)$ is connected and of genus 0, and

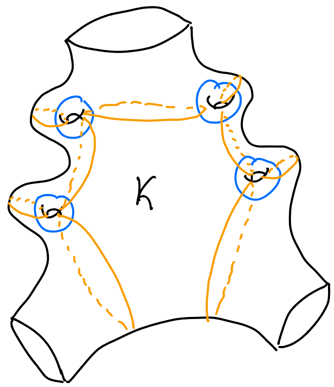

FIGURE 2. Curves in a Lickorish generating set for $\mathrm{MCG}(K)$.

(iv) the intersection of the support of h_i with K has genus 2,

where h_i^+ is the attracting end of h_i and h_i^- is the repelling end. Up to homeomorphism, the $\{h_i\}_{i=1}^{n-1}$ and K are as in Figure 1 (in the 3-ended case).

For distinct $i, j \in \{1, \ldots, n\}$ choose a mapping class σ_{ij} such that $\sigma_{ij}(K) = K$, $\sigma_{ij}(e_i) = e_j$, $\sigma_{ij}(e_j) = e_i$, and $\sigma_{ij}(e_k) = e_k$ for $k \neq i, j$.

Now we choose a Lickorish generating set for the mapping class group of K that is suited to our choice of handle shifts. To avoid a notational mess, let A denote the set of non-separating simple closed curves in K represented in Figure 2. It is well-known that the right and left Dehn twists about the curves in A generate the (pure) mapping class group of K.

Define

$$F = \{h_i, h_i^{-1} : i \in \{1, \ldots, n-1\}\} \cup \{T_a, T_a^{-1} : a \in A\} \cup \{\sigma_{ij} : i, j \in \{1, \ldots n\} \& i \neq j\},$$

where T_a is the (left) Dehn twist about the curve a. We leave it as an exercise for the reader to verify that the subgroup generated by F contains all compactly supported mapping classes, i.e. the mapping classes with a representative that is the identity outside of a compact set.

We can view $\mathcal{C}(K)$ as a subset of $\mathcal{C}(S)$ and we define

$$U = U_K = \{f \in \mathrm{MCG}(S) : f(a) = a \text{ for all } a \in \mathcal{C}(K) \subset \mathcal{C}(S)\}.$$

We can show that $\mathcal{S} = U \cup F$ generates $\mathrm{MCG}(S)$: Let $f \in \mathrm{MCG}(S)$ and let K' be a connected, compact subsurface of S containing $K \cup f(K)$. Let $c_1, \ldots, c_n$ be a labeling of the components of ∂K such that $\mathrm{supp}(h_i) \cap c_j \neq \emptyset$ if and only if $j \in \{i, n\}$. By the classification of compact surfaces, there exists a mapping class f_1 supported in K' such that for every $i \in \{1, \ldots n\}$ there exists $j_i \in \{1, \ldots, n\}$ and $k_i \in \mathbb{Z}$ such that $f_1 \circ f(c_i) = h^{k_i}(c_{j_i})$ for some $k_i \in \mathbb{Z}$.

If

$$f_2 = \left(\prod_{i=1}^{n-1} h^{-k_i} \right) f_1,$$

then $f_2 \in \langle F \rangle$ and $f_2 \circ f(c_i) = c_{j_i}$ for every $i \in \{1, \dots, n\}$. Let σ be the permutation of $\{1, \dots, n\}$ defined by $\sigma(i) = j_i$. We can choose a mapping class f_σ written as a product of the σ_{ij} realizing the permutation.

Now, as $f_\sigma \circ f_2 \circ f$ fixes every component of ∂K, there exists a mapping class f_3 supported in K such that $f_3 \circ f_\sigma \circ f_2 \circ f$ is in U; hence, $f \in \langle U \cup F \rangle$.

By Lemma 14, U is coarsely bounded and hence $\mathcal{S} = U \cup F$ is coarsely bounded as F is finite. We can conclude that $\mathrm{MCG}(S)$ is CB generated. □

Corollary 15. *Let S is an infinite-genus surface with a finite number of ends, none of which are planar. If $\mathcal{S}$ is the CB generating set constructed in Theorem 13, then $\mathcal{S}$ is analytic and hence every left-invariant compatible metric on* $\mathrm{MCG}(S)$ *is quasi-isometric to the word metric associated to $\mathcal{S}$.*

Proof. Let U and F be as in the proof of Theorem 13. Then, $\mathcal{S} = U \cup F$ is closed as U was closed and F was finite. Every closed subset of a Polish space is Polish and hence $\mathcal{S}$ is Polish and hence analytic. The remainder of the corollary follows from Theorem 12. □

Remark 16. It is not too difficult to see that Theorem 13 and Lemma 14 can be readily extended to planar surfaces whose end space is homeomorphic to an ordinal space of the form $\omega \cdot n + 1$ for $n \in \mathbb{N}$. However, in general, it is not clear how to extend the proofs given above to other surfaces; the proofs appear to fundamentally rely on having a finite number of ends (or in the planar case, a finite number of non-isolated ends).

We have now established an infinite family of big mapping class groups with a well-defined quasi-isometry class. For each $n \in \mathbb{N}$, let L_n denote the infinite-genus surface with n ends, none of which are planar. We can now ask the following:

Is $\mathrm{MCG}(L_n)$ *quasi-isometric to* $\mathrm{MCG}(L_m)$ *if and only if* $n \neq m$*?*

(This question was asked by Kasra Rafi at the AIM workshop.)

As a first case, we have:

Proposition 17. *If $n \in \mathbb{N}$ such that $n \geq 2$, then* $\mathrm{MCG}(L_1)$ *and* $\mathrm{MCG}(L_n)$ *are not quasi-isometric.*

Proof. As noted earlier, $\mathrm{MCG}(L_1)$ is coarsely bounded [14]. In contrast, there is a continuous homomorphism from a finite-index subgroup of $\mathrm{MCG}(L_n)$ onto $\mathbb{Z}$ [1, Corollary 2], which implies that $\mathrm{MCG}(L_n)$ is not coarsely bounded. (Note: it is also shown that $\mathrm{MCG}(L_n)$ is not coarsely bounded in [14] and, for $n \geq 4$, it is also shown in [7, Corollary 1.3].) □

We finish this note by giving an explicit construction of a graph quasi-isometric to $\mathrm{MCG}(L_2)$. Let Γ be the graph whose vertices correspond to homotopy classes of homologically non-trivial separating curves in L_2 and where two vertices are adjacent if they can be realized by disjoint curves co-bounding a genus-1 subsurface. This graph was first introduced in [7, Section 9].

In an earlier version of these notes, the conjecture below was claimed as a theorem. However, it was pointed out to me by Anschel Schaffer-Cohen that my proof was incomplete. After communicating with Schaffer-Cohen, I believe he has a proof of the the conjecture and that it will appear in a forthcoming paper.

Conjecture 18. $\mathrm{MCG}(L_2)$ *and* Γ *are quasi-isometric.*

In [7, Proposition 9.6] it is shown that Γ is not hyperbolic and so, if the conjecture holds, it would follow that $\mathrm{MCG}(L_2)$ is not hyperbolic.

Appendix A. Topology of homeomorphism groups

The goal of this section is to give an alternative proof that mapping class groups are Polish via studying the compact-open topology on the group of homeomorphisms. Let S be a surface and let $\mathrm{Homeo}(S)$ denote the group of self-homeomorphisms of S. We consider $\mathrm{Homeo}(S)$ equipped with the compact-open topology, that is, the topology generated by sets of the form

$$V(K,U) = \{f \in \mathrm{Homeo}(S) : f(K) \subset U\}.$$

The compact-open topology was introduced simultaneously and independently by Fox [10] and Arens [3]. (The definitions hold more generally for function spaces with locally-compact or first-countable domains; also, an equivalent topology for homeomorphism groups goes back to at least Birkhoff [6].) The group $\mathrm{Homeo}(S)$ equipped with the compact-open topology is a topological group [2, Theorem 3]. (It is not so hard to see this for compact Hausdorff spaces; Arens then uses the Alexandroff compactification to extend it to locally-compact Hausdorff spaces.)

Let us explain why $\mathrm{Homeo}(S)$ is Polish. As S is second countable and Hausdorff, it is an exercise to show that $\mathrm{Homeo}(S)$ is also second countable and Hausdorff. In particular, by the Birkhoff-Kakutani theorem, $\mathrm{Homeo}(S)$ is metrizable. Let's consider an explicit metric: Fix a complete metric d on S. If S is compact, then define $\rho\colon \mathrm{Homeo}(S)^2 \to \mathbb{R}$ by

$$\rho(f,g) = \max_{x\in S}\{d(f(x),g(x))\}.$$

Now suppose that S is not compact. Fix a compact exhaustion $\{K_n\}_{n\in\mathbb{N}}$ of S, that is, K_n is compact, K_n is contained in the interior of K_{n+1}, and $S = \bigcup_{n\in\mathbb{N}} K_n$. For $f, g \in \mathrm{Homeo}(S)$ define

$$\delta_n(f,g) = \min\{\max\{f(x), g(x) : x \in K_n\}, 2^{-n}\}.$$

Now define $\rho(f,g) = \sum_{n\in\mathbb{N}} \delta_n(f,g)$. (Note that $\rho(f,g) \leq 1$ for all $f, g \in \mathrm{Homeo}(S)$.) This is the metric constructed in [3, Theorem 7] (with the additional requirement that d be complete). Arens proves that the metric topology corresponding to ρ is the compact-open topology.

As the inversion map $f \mapsto f^{-1}$ is a homeomorphism of $\mathrm{Homeo}(S)$, we have that $\mu(f,g) = \rho(f,g) + \rho(f^{-1},g^{-1})$ defines a metric on $\mathrm{Homeo}(S)$ compatible with the compact-open topology. In fact, μ is a complete metric on $\mathrm{Homeo}(S)$. To see this, observe that given a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $(\mathrm{Homeo}(S), \mu)$ we can define the map $f\colon S \to S$ by $f(x) = \lim f_n(x)$ – the limit exists as the sequence $\{f_n(x)\}$ is Cauchy in S with respect to d, which is complete. We leave it as exercise to show that $f \in \mathrm{Homeo}(S)$ and hence μ is complete. (As an aside, we note that the metric

ρ is not complete; it is critical that the sequence $\{f_n^{-1}\}$ is Cauchy whenever $\{f_n\}$ is Cauchy.) We have shown:

Proposition 19. *For every surface S, the group* Homeo(S) *is Polish.* □

Let $\mathrm{Homeo}_0(S)$ denote the path component of the identity in Homeo(S). We can then equip $\mathrm{MCG}(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$ with the quotient topology. Using the Alexander method [9, Proposition 2.8], it is not difficult to see that this quotient topology agrees with the permutation topology defined earlier.

Proposition 20. $\mathrm{Homeo}_0(S)$ *is a closed normal subgroup of* Homeo(S) *for every surface S*

Proof. We have already seen that MCG(S) is Hausdorff by Proposition 2. The proposition follows from the standard fact that a quotient of a topological group is Hausdorff if and only if the kernel is closed. □

Corollary 21. *Mapping class groups are Polish.*

Proof. The quotient of a Polish group by a closed normal subgroup is a Polish group (see [5, Proposition 1.2.3] and [13, Theorem 8.19]). □

Note that this proof does not give the stronger result of Corollary 6.

Above, to see that $\mathrm{Homeo}_0(S)$ is closed, we relied on the identification of the quotient of the compact-open topology with the permutation topology; if the reader prefers to avoid this, we prove below that $\mathrm{Homeo}_0(S)$ is in fact the entire connected component of the identity and hence is closed.

Before doing so, we state the following theorem of Epstein that establishes the bijection between homotopy classes and isotopy classes of homeomorphisms.

Theorem 22 ([8, Theorem 6.4])**.** *Let S be a surface not homeomorphic to either the plane or the annulus. If $f \in$* Homeo(S) *is homotopic to the identity, then it is isotopic to the identity.*

Note: if S is the plane or the annulus, then every orientation-preserving homeomorphism homotopic to the identity is isotopic to the identity. We should also note that for compact surfaces, Theorem 22 goes back to Baer. Combining Theorem 22 with [10, Theorem 1], we immediately obtain:

Proposition 23. *Let S be a surface not homeomorphic to either the plane or the annulus. A homeomorphism $f \in$* Homeo(S) *is isotopic to the identity if and only if $f \in \mathrm{Homeo}_0(S)$.* □

We can now prove that $\mathrm{Homeo}_0(S)$ is the connected component of the identity in Homeo(S). The idea of the proof comes from Mladen Bestvina as communicated to the author by Jing Tao.

Proposition 24. *For every surface S, the connected component of the identity in* $\mathrm{Homeo}(S)$ *is equal to* $\mathrm{Homeo}_0(S)$.

Proof. For each simple closed curve γ on S, let U_γ be a regular neighborhood of γ. Note that if $f \in V(\gamma, U_\gamma)$, then $f(\gamma)$ is isotopic to γ.

Suppose S is of finite type. By the Alexander method [9, Proposition 2.8], there exists a finite collection $\gamma_1, \dots, \gamma_n$ of simple closed curves on S such that

$$\mathrm{Homeo}_0(S) = \bigcap_{i=1}^{n} V(\gamma_i, U_{\gamma_i}).$$

In particular, $\mathrm{Homeo}_0(S)$ is open. As $\mathrm{Homeo}(S)$ is a disjoint union of translates of $\mathrm{Homeo}_0(S)$, we can conclude that $\mathrm{Homeo}_0(S)$ is also closed and hence equal to the connected component of the identity.

We can now assume that S is of infinite type. Fix a metric d on S and a compact exhaustion $\{K_n\}_{n\in\mathbb{N}}$. Using d and the exhaustion $\{K_n\}$, construct the metric μ on $\mathrm{Homeo}(S)$ as above as in the proof of Proposition 19.

Let C denote the connected component of the identity in $\mathrm{Homeo}(S)$ and let B_δ denote the intersection of the δ-ball (with respect to μ) with C. Recall that in a connected topological group, every neighborhood of the identity generates the group. Therefore, B_δ generates C.

Fix a simple closed curve γ on S. For every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for every $x \in \gamma$, $d(x, g(x)) < \epsilon$ whenever $g \in B_\delta$. In particular, there exists ϵ_γ such that whenever $g \in B_{\delta_\gamma}$, we have $g(\gamma) \subset U_\gamma$ and hence $g(\gamma)$ is isotopic to γ, where $\delta_\gamma = \delta(\epsilon_\gamma)$.

Now let $f \in C$, then we can write $f = g_1 \cdots g_k$ with $g_j \in B_{\delta_\gamma}$ and hence $f(\gamma)$ is isotopic to γ. In particular, f fixes the isotopy class of every simple closed curve in S; hence, by [11, Corollary 1.2], f is isotopic to the identity and thus $f \in \mathrm{Homeo}_0(S)$ by Proposition 23. □

The proof of Proposition 24 is very special to two dimensions, which leads to the question for arbitrary dimensions:

> *If M is a second-countable manifold, is the connected component of the identity in* $\mathrm{Homeo}(M)$ *path connected?*

It is possible that this is well known, but the author is unaware of a reference.

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