INFINITE-TYPE SURFACES AND MAPPING CLASS GROUPS: OPEN PROBLEMS

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Abstract. These are notes on the open problem session run by Priyam Patel and Nicholas Vlamis for the infinite-type surfaces group at the 2021 Nearly Carbon Neutral Geometric Topology conference organized by Elizabeth Field, Hannah Hoganson, and Marissa Loving. The notes have been typed by Yassin Chandran.

1. Introduction

The goal of the problem session was to introduce the audience to themes in the current research of mapping class groups of infinite-type surfaces. To that end, this is by no means meant to be an exhaustive list, and in fact, many of the questions are purposefully vague. During the month between Patel and Vlamis's first discussion and the presentation, several of their initial questions were answered. The body of literature in the field is growing quickly, and so the reader is encouraged to investigate recent progress before embarking on any specific question.

For a general introduction to infinite type mapping class groups see [8]. This note will be divided into the following three main categories:

(1) Algebraic

(2) Teichmüller Theory and Actions on Complexes

(3) Geometric Group Theory and Topology

All surfaces considered are connected, orientable, and second countable.

2. Algebraic Questions

2.1. **Generating mapping class groups of infinite-type surfaces.** Patel and Vlamis in [34] showed that pure mapping class groups of infinite-type surfaces are *topologically generated* by Dehn twists and handle shifts. However, if one is studying homomorphisms, then, without a continuity assumption, topological generators do not provide enough information about the group.

Question 2.1. *Give a "nice" set of elements that algebraically generate* $\mathrm{MCG}(S)$ *(or* $\mathrm{PMCG}(S)$*).*

Remark. Note that for an infinite-type surface any such collection is necessarily uncountable since $\mathrm{MCG}(S)$ is itself uncountable.

Malestein and Tao [27] give algebraic generating sets consisting of involutions for a surfaces with zero or infinite genus and whose end space is self-similar and contains a Cantor set of maximal end. These surfaces are called *uniformly self-similar*.

Calegari and Chen in [15] showed that if S is a finite-type surface minus a Cantor set and with at least one isolated planar end, then $\mathrm{MCG}(S)$ is generated by torsion elements. This is a corollary of their investigation of normal subgroups.

Mann and Rafi in [29] have some results concerning *coarsely bounded* (CB) generating sets.

Given the importance of Dehn twists as generators in the finite-type setting, it is natural to ask:

Question 2.2. *What group is generated by the set of (infinite) multi-twists?*

2.2. **Algebraic invariants and subgroups.** If a surface has finite genus, then the mapping class group surjects onto the mapping class group of the closed surface of the same genus. One can then pullback any subgroup of the mapping class group of the closed surface to that of the original surface. In this way, using the theory of mapping class groups of finite-type surfaces, it is easy to construct large subgroups of mapping class groups of infinite-type surfaces with positive finite genus. However, outside of this setting, it has been difficult to find large (e.g. finite- or countable-index) subgroups of mapping class groups of infinite-type surfaces.

With this discussion in mind, for the rest of this subsection, we restrict ourselves to infinite-genus surfaces with no planar ends. (There are likely versions for the genus zero case and for allowing some planar ends.)

Question 2.3. *Given a finite- or countable-index subgroup of a mapping class group is it necessarily a pullback of a finite- or countable-index subgroup of either the homeomorphism group of the end space or the abelianization of the mapping class group?*

Unlike the finite-type case, the abelianization of (pure) mapping class groups of infinite-type surfaces can be large, both in some natural ways [7] and some surprising ways [19]. In the latter case, it was shown the the mapping class group of the Loch Ness monster surface has nontrivial abelianization, and in the abelianization there are many copies of $\mathbb{Q}$ and hence the mapping class group of the Loch Ness monster surface has many proper subgroups of countably infinite index.

In contrast, it was recently shown in [24] that the abelianization of infinite-type surfaces can be finitely generated and even discrete when equipped with the quotient topology.

Question 2.4. *Does the mapping class group of the Loch Ness monster surface have any proper finite-index subgroups? Does the abelianization have any torsion?*

More generally, one may ask about simply constructing and classifying subgroups of mapping class groups. In a similar vein, one can find and classify the actions of mapping class groups. For instance, a classical example is the action of the mapping class group of a surface on the first homology group of the surface and its kernel, the Torelli group (see for instance [22] and [6]).

Problem 2.5. *Find methods of constructing subgroups/actions of mapping class groups.*

Towards this goal is the recent work of Abbott, Hoganson, Loving, Patel, and Skipper [2]. A group G is called *indicable* if there exists a surjective homomorphism $f : G \to \mathbb{Z}$. For any indicable group G that arises as a subgroup of the mapping class group of a surface with one boundary component, the authors use shift maps to construct uncountably many embeddings of G into a big mapping class group $\mathrm{MCG}(S)$; moreover, the images of G are not contained in the closure of set of compactly supported mapping classes (and hence are "intrinsically infinite-type").

Next, we turn to finding small (i.e. countable) subgroups of mapping class groups.

Question 2.6. *Does the mapping class group of an infinite-genus surface with no planar ends contain every countable group?*

The work of Aougab, Patel, and Vlamis [4] show that the above question is true for the infinite-genus surfaces with no planar ends and self-similar end space, e.g. the Loch Ness monster surface; it is also shown that every mapping class group of an infinite-genus surface with no planar ends contains every finite group as a subgroup. To do this, they realize every countable group as the isometry group of a hyperbolic structure, and moreover, show that the above surfaces are the only ones in which this is possible. Therefore, to answer Question 2.6, it is necessary to construct infinitely countable subgroups in a new way.

Remark. There are restrictions on the countable groups that can appear as subgroups once you relax the topological restrictions—see [4, Section 9] for more details.

Next, we turn to quotients. Every finite group can be realized as a quotient of a finite-index subgroup of a mapping class group of some closed surface.

Question 2.7. *Does there exist a finite (or countable) group that is not the quotient of a (finite-index subgroup of a) mapping class group of an infinite-genus surface?*

Question 2.8. *Does there exist a single mapping class group for which every countable (or finite) group is a quotient? Equivalently (in the countable case), is there a mapping class group that surjects onto the free group on a countable set?*

The following is a question of Bestvina from the AIM work shop on infinite-type surfaces [1].

Question 2.9. *Let S be the plane minus a cantor set. Is it true that every subgroup of* $\mathrm{MCG}(S)$ *either has an infinite-dimensional space of quasimorphisms or is amenable?*

3. Teichmüller Theory and Actions on Complexes

The problem motivating much of this section is to generalize the Nielsen–Thurston classification of surface homeomorphisms, that is:

Problem 3.1. *Classify homeomorphisms of infinite-type surfaces.*

As a sub-problem, it is natural to ask:

Question 3.2. *What is the correct analogue (or analogues) of pseudo-Anosov homeomorphisms in the infinite-type setting?*

3.1. Teichmüller Space.

The Teichmüller space, $\mathrm{Teich}(R)$, of a Riemann surface R is the set of equivalence classes of pairs (X, f) where $f\colon R \to X$ is a quasiconformal homeomorphisms and where $(X, f) \sim (Y, g)$ if $g \circ f^{-1}\colon X \to Y$ is homotopic to a conformal map. Regardless of whether R is of finite or infinite type, the Teichmüller metric is defined the same way, and defines a topology on $\mathrm{Teich}(R)$ (in the case R is of infinite type, there are several "natural" topologies on $\mathrm{Teich}(R)$ that can be defined and may not always agree). The Teichmüller modular group, denoted $\mathrm{Mod}(R)$, is the group of homotopy classes of quasiconformal homeomorphisms $R \to R$. From the definitions, one sees that $\mathrm{Mod}(R)$ acts on $\mathrm{Teich}(R)$. We readily see that $\mathrm{Mod}(R) < \mathrm{MCG}(R)$, but if R is an infinite-type surface, then it is the case that $\mathrm{MCG}(R) \neq \mathrm{Mod}(R)$, and moreover, $\mathrm{MCG}(R)$ does not act on $\mathrm{Teich}(R)$ (this currently is not written in the literature anywhere). Note that in the infinite-type setting, the group $\mathrm{Mod}(R)$ has a much longer history of being studied.

In the finite-type setting, the action of mapping class groups on Teichmüller space is crucial ingredient in the celebrated Nielsen–Thurston classification of mapping classes. This motivates the following few questions, the first of which is quite vague, but nonetheless is worth thinking about.

Question 3.3. *How does* $\mathrm{Mod}(R)$ *act on* $\mathrm{Teich}(R)$*?*

In Bers's proof of the Nielsen–Thurston classification, the dynamics of a single mapping class on Teichmüller space fall into three categories, which yields the classification theory; however, the picture is not as clean in the infinite-type setting—we refer the reader to [32]. Thurston's original proof relied on the boundary of Teichmüller space and the compactness of the associated compactification. In the infinite-type setting, it is still possible to discuss analogous boundaries (see Bonahon–Šarić [14]), but they will not yield a compactification.

Question 3.4. *How does* $\mathrm{Mod}(R)$ *act on the boundary of* $\mathrm{Teich}(R)$*?*

This problem session is interested in the full mapping class group, so to get back on this track, we propose the following:

Question 3.5. *How does* $\mathrm{Mod}(R)$ *sit in* $\mathrm{MCG}(R)$*?*

For instance, one can modify a construction of Matsuzaki [31] (and using [9] to guarantee completeness) to see that it is possible to have $\mathrm{Mod}(R)$ be equal to the group of compactly supported mapping class, and in particular, for $\mathrm{Mod}(R)$ to be a normal subgroup of $\mathrm{MCG}(R)$.

Problem 3.6. *Characterize when* $\mathrm{Mod}(R)$ *is normal in* $\mathrm{MCG}(R)$*.*

Note that $\mathrm{Mod}(R)$ can be dense in $\mathrm{MCG}(R)$, and hence, even though $\mathrm{MCG}(R)$ does not act on $\mathrm{Teich}(R)$, it may be possible to import information from the action of $\mathrm{Mod}(R)$ of $\mathrm{Teich}(R)$. (For example, if R is homeomorphic to the Loch Ness monster surface, then the group of compactly supported mapping classes is dense in $\mathrm{MCG}(R)$, and hence $\mathrm{Mod}(R)$ is dense in $\mathrm{MCG}(R)$ since every compactly supported mapping class is quasiconformal.)

3.2. **Action on complexes.** In the finite-type setting, the action of mapping class groups on the curve graph as has been an incredibly fruitful tool. This graph is infinite diameter and Gromov hyperbolic [30].

In the infinite-type setting, the curve graph is diameter 2 and thus quasi-isometrically trivial. However, there have been a number of papers constructing infinite-diameter hyperbolic graphs upon which infinite-type mapping class groups act, see [5, 10, 12, 13, 20, 21, 25, 38] (n.b. [11] is an English translation of [10]).

Problem 3.7. *Study actions of* $\mathrm{MCG}(S)$ *on infinite-diameter hyperbolic graphs.*

In the finite-type setting, pseudo-Anosov mapping classes act loxodromically on the curve graph. In reference to Question 3.2, we want to understand the mapping classes that act loxodromically on the various hyperbolic graphs referenced above.

Question 3.8. *Construct and classify loxodromics for the various actions referenced above.*

Bavard [10] was the first to construct loxodromic actions, and recently Morales–Valdez [33] and Abbott–Patel–Miller [3] have given additional new constructions.

In the finite-type setting, no mapping classes act parabolically on the curve graph:

Question 3.9. *Do there exist mapping classes that act parabolically on any of the referenced graphs? If so, classify them.*

Before the recent interest in mapping class groups of infinite-type surfaces, homeomorphisms of infinite-type surfaces were studied in the context of foliations on 3-manifolds. In this body of work, the notion of *end periodic homeomorphism* plays a crucial role and there is a Nielsen–Thurston classification for such homeomorphisms—we recommend the reader see the recent work [23, Section 2.2] for a short introduction, including definitions and references, for end periodic homeomorphisms (we recommend this recent paper because it was written with the work on big mapping class groups in mind). In fact, end periodic homeomorphisms have properties in common with pseudo-Anosov homeomorphisms, and it is natural to ask:

Question 3.10. *When do end periodic homeomorphisms act loxodromically on the above referenced graphs?*

We also want to bring attention to the work of Šarić on the theory of train tracks and laminations for infinite-type surfaces [37], which is a useful tool when thinking about the questions above.

4. Geometric Group Theory and Topology

Geometry. Geometric group theory is most commonly applied to finitely generated groups, but much of theory can be adapted to the setting of locally compact, compactly generated groups, see [18]. Rosendal [36] observed that the compactness conditions can be relaxed by replacing compactness with a notion of boundedness and has extended the tools of geometric group theory to a larger class of groups, namely CB generated Polish groups. Mapping class groups of infinite-type surfaces fail are neither locally compact nor compactly generated; however, Mann and Rafi [29] characterize which mapping class groups are CB generated—and hence have a canonical (up to quasi-isometry) word metric—and, in particular, show that CB generated mapping class groups exist.

In fact, there are mapping class groups of infinite-type surfaces that are Gromov hyperbolic [38], a phenomenon that does not exist in the finite-type setting. The panelists are not familiar with the theory of locally compact, compactly generated groups in a meaningful way, but there is work focused on hyperbolic locally compact groups. We therefore ask a very generic question:

Question 4.1. *Does hyperbolicity imply any algebraic properties of (non-locally compact) CB generated Polish groups?*

And if the above question can be answered in the affirmative, it is natural to investigate further:

Problem 4.2. *Develop the theory of hyperbolic Polish groups.*

Automatic continuity. Mapping class groups are Polish groups. For a comprehensive reference on Polish groups see [26]. A Polish group G has the automatic continuity property (ACP) if every group homomorphism $G \to H$ is continuous whenever H is a separable topological group. For example, the homeomorphism group of the Cantor set, the homeomorphism groups of closed manifolds, and the automorphism group of the countably infinite-rank free group all have the ACP (see [35] for a survey and references regarding ACP).

Mann [28] has given examples of mapping class groups with and without the automatic continuity property. Domat and Dickmann [19] have shown that the mapping class group of the Loch Ness monster surface fails to have the automatic continuity property by constructing a discontinuous homomorphism to the rationals (the ideas behind this case have been generalized in [27] to provide additional examples of mapping class groups without the ACP).

Problem 4.3. *Characterize the mapping class groups with the ACP.*

The known examples of discontinuous homomorphisms from mapping classes each have either a finite group or the rationals as their codomain. This does not seem to be a coincidence given Conner's conjecture [16]: a group H is *cm-slender* if every homomorphism with domain a completely metrizable group and with H as a codomain has an open kernel.

Conjecture 4.4 (Conner's Conjecture)**.** *Every countable torsion-free group that does not contain an isomorphic copy of $\mathbb{Q}$ is cm-slender.*

Conner's conjecture is known to hold for a large class of groups, including torsion-free word hyperbolic groups, free abelian groups, braid groups, and Thompson's groups (see [17]).

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