MATH 201: Fall 2025 **Instructor: Nicholas Vlamis**

Wednesday 10/15/2025 Exam 1 110 minutes

Name: Solutions

Instructions.

1. ***Read each problem carefully.*** Make sure you understand the problem.
2. Unless otherwise specified, **you must show all your work.** No credit will be given to a problem without work. Feel free to write on the back of any page, but please make it clear where your work and answers are.
3. Unless previously granted permission, you may only use a TI-82, TI-83, TI-84 or scientific calculator.
4. You may use a note sheet, which consists of a single sheet of 8.5" x 11" inch paper. Your note sheet is **not** allowed to contain solutions to problems or proofs of theorems. It will be collected with your exam.
5. No devices other than a writing utensil and calculator may be used.
6. Unless otherwise noted, answers must be precise, not approximate (eg, $\frac{1}{3} \neq 0.33$).

Question	Points	Score
1	7	
2	4	
3	7	
4	6	
5	4	
6	10	
7	7	
8	5	
Total:	50	

1. $\boxed{\text{7 points}}$ Consider the plane curve parameterized by $C(t) = (t^3 - 6t^2 + 9t, t^3 - 3t)$ for all t in $(-\infty, \infty)$. Find all the values of t for which C has (i) horizontal tangent(s) and (ii) vertical tangent(s).

$$\frac{dx}{dt} = 3t^2 - 12t + 9 \quad , \quad \frac{dy}{dt} = 3t^2 - 3$$

$$\Rightarrow \frac{dy}{dx} = \frac{3t^2-3}{3t^2-12t+9} = \frac{t^2-1}{t^2-4t+3}$$

$$= \frac{(t-1)(t+1)}{(t-1)(t-3)}$$

C has a horizontal tangent at $t=a$ if $\lim_{t \to a} \frac{dy}{dx} = 0$ and a vertical tangent if $\lim_{t \to a} \frac{dy}{dx} = \infty$.

$\Rightarrow$ C has a horizontal tangent at $t = -1$ and a vertical tangent at $t = 3$.

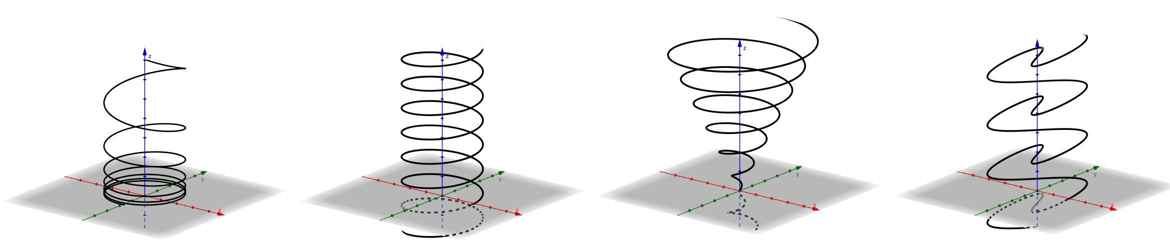
2. $\boxed{\text{4 points}}$ Match each of the curves in the figures with one of the following parameterizations (no explanation necessary).

(C) (i) $(\frac{t}{2}\cos(10t), \frac{t}{2}\sin(10t), t)$

(A) (ii) $(\cos(10t), \sin(10t), e^{-t})$

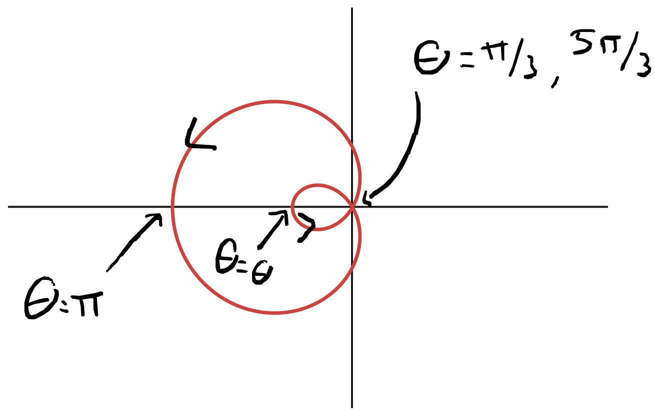
(D) (iii) ~~$(\cos(5t), \sin(5t), t)$~~ $(\cos(5t), \sin(10t), t)$

(B) (iv) $(\cos(10t), \sin(10t), t)$



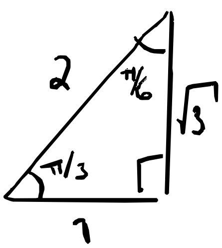
(A) (ii) | (B) (iv) | (C) (i) | (D) (iii)

3. $\boxed{\text{7 points}}$ Consider the limaçon with polar equation $r(\theta) = 1 - 2\cos(\theta)$, whose graph is shown below.



(a) On the graph, on the inner loop and on the outer loop, draw an arrow indicating which way the curve is traversed as θ increases.

(b) Find the values of θ between 0 and 2π for which the limaçon intersects the x-axis, and label the intersection points on the graph with the corresponding values of θ.



$y(\theta) = (1-2\cos\theta)\sin\theta$ Need to solve $y(\theta)=0$.

$y(\theta)=0 \Rightarrow 1-2\cos\theta=0$ or $\sin\theta=0$

$1-2\cos\theta=0$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \pi/3,\ 5\pi/3$

$\sin\theta = 0$

$\Rightarrow \theta = 0,\ \pi$

Now,

$x(0)=-1,\ x(\pi)=-2$

$x(\pi/3)=x(5\pi/3)=0$

(c) Setup, but **do not evaluate**, a definite integral to compute the area enclosed by the inner loop of the limaçon. Explain your choice of limits in the integral.

There are a couple of options. The tricky part is to realize that there is no interval in 0 to 2π for which we traverse only the inner loop.

1) $\displaystyle\int_{-\pi/3}^{\pi/3} \frac{1}{2}(1-2\cos\theta)^2\,d\theta$ 2) $\displaystyle\int_{5\pi/3}^{7\pi/3} \frac{1}{2}(1-2\cos\theta)^2\,d\theta$

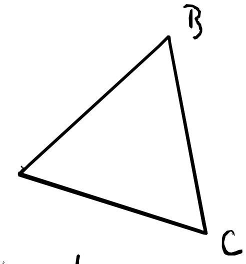
3) Use the symmetry about the x-axis: $\displaystyle 2\int_0^{\pi/3} \frac{1}{2}(1-2\cos\theta)^2\,d\theta$

4. $\boxed{\text{6 points}}$ Consider the points $A = (0, 1, 1)$, $B = (1, 0, 1)$, and $C = (1, 1, 0)$.

(a) Find the equation of the plane (in the form $ax+by+cz+d = 0$) that passes through the points A, B, and C.

$\vec{AB} = \langle 1, -1, 0 \rangle$

$\vec{AC} = \langle 1, 0, -1 \rangle$

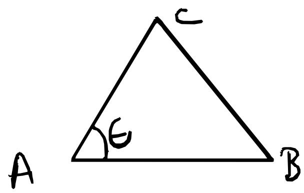


Both $\vec{AB}$ and $\vec{AC}$ are in the direction of the plane, so $\vec{n} = \vec{AB} \times \vec{AC}$ is a normal vector for the plane.

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

$\Rightarrow$ The equation of the plane is $(x-0)+(y-1)+(z-1) = 0$ or $x+y+z-2=0$.

(b) Find the area of the triangle whose vertices are A, B, and C.

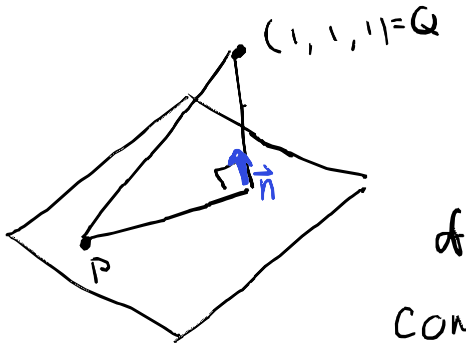


$$\text{Area} = \frac{1}{2} |\vec{AB}||\vec{AC}| \sin\theta = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{3}$$

5. $\boxed{\text{4 points}}$ Find the distance between the point $(1, 1, 1)$ and the plane with equation $3x + 2y + 6z = 5$.

$\vec{n} = \langle 3, 2, 6 \rangle$ is a normal vector to the plane.

The point $P = (\frac{5}{3}, 0, 0)$ is on the plane



The distance from $Q = (1,1,1)$ to the plane is scalar projection of $\vec{PQ}$ onto $\vec{n}$:

$$\text{comp}_{\vec{n}} \vec{PQ} = \frac{\langle -2/3, 1, 1 \rangle \cdot \langle 3, 2, 6 \rangle}{|\langle 3, 2, 6 \rangle|} = \frac{6}{7}$$

6. $\boxed{\text{10 points}}$ Let C be the space curve with vector function $\mathbf{r}(t) = \langle t^2 - t, t^3 - 2t^2, t + \cos t\rangle$.

(a) Compute the curvature of C at $t = 0$.

We'll use the formula $k(t) = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$r'(t) = \langle 2t-1, 3t^2-4t, 1-\sin t\rangle$

$r''(t) = \langle 2, 6t-4, -\cos t\rangle$

$r'(0) = \langle -1, 0, 1\rangle$

$r''(0) = \langle 2, -4, -1\rangle$

$|r'(0)| = \sqrt{2}$

$r'(0) \times r''(0) = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 2 & -4 & -1 \end{vmatrix}$

$= \langle 4, 1, 4\rangle$

$$\boxed{k(0) = \frac{\sqrt{33}}{\sqrt{8}} = \frac{1}{2}\sqrt{\frac{33}{2}}}$$

(b) Let $\mathbf{T}(t)$ be the unit tangent vector to C. The unit normal vector $\mathbf{N}(t)$ can be computed using the following formula:

$$\cancel{\mathbf{N} = \frac{\mathbf{r}'' - (\mathbf{r}'' \cdot \mathbf{T}) \cdot \mathbf{T}}{|\mathbf{r}'' - (\mathbf{r}'' \cdot \mathbf{T}) \cdot \mathbf{T}|}}. \qquad N = \frac{r'' - (r'' \cdot T)T}{|r'' - (r'' \cdot T)T|}$$

Find $\mathbf{N}$ when $t = 0$.

$T(t) = \dfrac{r'(t)}{|r'(t)|}$, $T(0) = \dfrac{\langle -1, 0, 1\rangle}{\sqrt{2}}$

$r''(0) - (r''(0) \cdot T(0))T = \langle 2, -4, -1\rangle - \left(\dfrac{-3}{\sqrt{2}}\right)\dfrac{\langle -1, 0, 1\rangle}{\sqrt{2}}$

$= \langle 2, -4, -1\rangle - \langle \frac{3}{2}, 0, -3/2\rangle = \langle \frac{1}{2}, -4, 1/2\rangle \Rightarrow N(0) = \dfrac{\langle \frac{1}{2}, -4, \frac{1}{2}\rangle}{\sqrt{33/2}}$

$$\Rightarrow \boxed{N(0) = \frac{1}{\sqrt{66}}\langle 1, -8, 1\rangle}$$

(c) Find the binormal vector $\mathbf{B}$ to C at $t = 0$.

$B(0) = T(0) \times N(0)$

$= \dfrac{1}{2\sqrt{33}}\left(\langle -1, 0, 1\rangle \times \langle 1, -8, 1\rangle\right)$

$= \dfrac{1}{2\sqrt{33}}\begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 1 & -8 & 1 \end{vmatrix} = \dfrac{1}{2\sqrt{33}}\langle 8, 2, 8\rangle = \dfrac{1}{\sqrt{33}}\langle 4, 1, 4\rangle$

$$\boxed{B(0) = \frac{1}{\sqrt{33}}\langle 4, 1, 4\rangle}$$

7. $\boxed{\text{7 points}}$ Find the position vector of a particle with initial position vector $\mathbf{r}_0 = \langle 0, 1, 0 \rangle$, initial velocity vector $\mathbf{v}_0 = \langle 1, 0, 0 \rangle$, and acceleration given by $\mathbf{a}(t) = \langle 2t, \sin t, \cos(2t) \rangle$.

$$v(t) = v_0 + \int_0^t \langle 2s, \sin s, \cos(2s) \rangle \, ds$$

$$= \langle 1, 0, 0 \rangle + \langle t^2, -\cos t + 1, \tfrac{1}{2}\sin(2t) \rangle$$

$$= \langle t^2 + 1, 1 - \cos t, \tfrac{1}{2}\sin(2t) \rangle$$

$$r(t) = r_0 + \int_0^t \langle s^2 + 1, 1 - \cos s, \tfrac{1}{2}\sin(2s) \rangle \, ds$$

$$= \langle 0, 1, 0 \rangle + \langle \tfrac{1}{3}t^3 + t, t - \sin t, -\tfrac{1}{4}\cos(2t) + 1/4 \rangle$$

$$= \langle \tfrac{1}{3}t^3 + t, t - \sin t + 1, \tfrac{1}{4}(1 - \cos(2t)) \rangle$$

8. $\boxed{\text{5 points}}$ Prove that if $\mathbf{r}(t)$ is a vector function with $|\mathbf{r}(t)|$ constant, then $\mathbf{r}$ and $\mathbf{r}'$ are orthogonal.

Given $r \cdot r$ is constant

$$\Rightarrow \frac{d}{dt} r \cdot r = 0$$

Now, $\frac{d}{dt} r \cdot r = r' \cdot r + r \cdot r' = 2 r \cdot r'$

$$\Rightarrow 2 r \cdot r' = 0$$

$$\Rightarrow r \cdot r' = 0$$

$\Rightarrow$ r and r' are orthogonal.