MATH 201: Fall 2025 **Instructor: Nicholas Vlamis**

Wednesday 11/19/2025 Exam 2 110 minutes

Name: Solutions

Instructions.

1. ***Read each problem carefully.*** Make sure you understand the problem.
2. Unless otherwise specified, **you must show all your work.** No credit will be given to a problem without work. Feel free to write on the back of any page, but please make it clear where your work and answers are.
3. Unless previously granted permission, you may only use a TI-82, TI-83, TI-84 or scientific calculator.
4. You may use a note sheet, which consists of a single sheet of 8.5" x 11" inch paper. Your note sheet is **not** allowed to contain solutions to problems or proofs of theorems. It will be collected with your exam.
5. No devices other than a writing utensil and calculator may be used.
6. Unless otherwise noted, answers must be precise, not approximate (eg, $\frac{1}{3} \neq 0.33$).

Question	Points	Score
1	6	
2	6	
3	8	
4	4	
5	8	
6	10	
7	8	
Total:	50	

1. $\boxed{\text{6 points}}$ Show that the following limit **does not exist**:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Consider the limit along the line $x=0$. Along this line:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1.$$

Now, along the line $y=0$, the limit becomes:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1.$$

As we get different results, $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ cannot exist

2. $\boxed{\text{6 points}}$ Let

$$f = \begin{cases} \frac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Is f continuous at $(0,0)$? Show all your work and explain your reasoning. (Hint: $x^2 \leq x^2 + y^2$ and $y^2 \leq x^2 + y^2$.)

For f to be continuous at $(0,0)$, we must have that $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$. Let's compute the limit using the squeeze theorem. For $(x,y) \neq (0,0)$

$$0 \leq \frac{x^2y^2}{x^2+y^2} \leq \frac{(x^2+y^2)(x^2+y^2)}{x^2+y^2} = x^2+y^2$$

As $\lim_{(x,y)\to(0,0)} x^2+y^2 = 0$, we have by the squeeze thm that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0 \Rightarrow f \text{ is continuous at } (0,0).$$

3. $\boxed{\text{8 points}}$ Let $F(x,y,z) = x^3 - xz^2 + 2e^y z$.

(a) Compute the partial derivatives F_x, F_y, F_z of F.

$F_x = 3x^2 - z^2$

$F_y = 2e^y z$

$F_z = -2xz + 2e^y$

(b) Find an equation for the tangent plane to the surface $F(x,y,z) = 1$ at the point $P(1,0,2)$.

$F_x(P) = -1$, $F_y(P) = 4$, $F_z(P) = -2$, so that equation of the tangent plane at P is

$(-1)(x-1) + 4(y-0) + (-2)(z-2) = 0$

or $-x + 4y - 2z = -5$

4. $\boxed{\text{4 points}}$ Suppose that f is a differentiable function of x and y, that x and y are differentiable functions of t, and that

$$\begin{array}{ll} x(2) = 3 & y(2) = -1 \\ x'(2) = 4 & y'(2) = 2 \\ f_x(3,-1) = 7 & f_y(3,-1) = 10 \end{array}$$

Use the above information to compute df/dt when $t = 2$. Make sure it clear how you arrived at your answer.

By the chain rule,

$$\left.\frac{df}{dt}\right|_{t=2} = \left.\frac{\partial f}{\partial x}\right|_{t=2} x'(2) + \left.\frac{\partial f}{\partial y}\right|_{t=2} y'(2)$$

$$= f_x(3,-1)(4) + f_y(3,-1)(2)$$

$$= (7)(4) + (10)(2)$$

$$= 48$$

5. $\boxed{\text{8 points}}$ Let $f(x, y)$ be differentiable near $(1, 2)$. Suppose that $\nabla f(1, 2) = \langle -1, 3\rangle$.

(a) Let $\mathbf{v} = \frac{1}{\sqrt{13}}\langle 2, -3\rangle$. Compute $D_v f(1, 2)$.

$$D_v f(1,2) = \nabla f(1,2) \cdot v = \langle -1, 3\rangle \cdot \frac{1}{\sqrt{13}}\langle 2, -3\rangle$$

$$= \frac{-11}{\sqrt{13}}$$

(b) Find the unit vector $\mathbf{u}$ maximizing $D_{\mathbf{u}} f(1, 2)$.

$D_u f(1,2)$ is maximized when

$$u = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{10}}\langle -1, 3\rangle$$

(c) Given your vector $\mathbf{u}$ from part (b), compute $D_{\mathbf{u}} f(1, 2)$.

$$D_u f(1,2) = |\nabla f| = \sqrt{10}$$

(d) Find the slope of the tangent line to the level curve of f through $(1, 2)$ at $(1,2)$. Explain the reasoning behind your answer.

The tangent line to the level curve of f through $(1,2)$ at $(1,2)$ is orthogonal to ∇f.

Therefore, the tangent line is in the direction of the vector $\langle 3, 1\rangle$, implying a slope of $1/3$.

6. $\boxed{\text{10 points}}$ Let $f(x, y) = 3x^3 - 3xy + y^2$.

(a) Find all the critical points of f.

$f_x = 9x^2 - 3y$

$f_y = -3x + 2y$

Need to solve $f_x = 0$ and $f_y = 0$ simultaneously.

$f_y = 0 \Rightarrow y = \frac{3}{2}x$

$\Rightarrow f_x = 9x^2 - \frac{9}{2}x$

Setting $f_x = 0$ yields $9x^2 - \frac{9}{2}x = 0$ or $x(x - 1/2) = 0$

$\Rightarrow x = 0$ or $x = 1/2$

$\Rightarrow (0,0)$ and $(1/2, 3/4)$ are the critical points of f.

(b) Classify each critical point as a local maximum, local minimum, or saddle point using the second derivative test.

We will use the second derivative test.

$$\left.\begin{aligned} f_{xx} &= 18x \\ f_{yy} &= 2 \\ f_{xy} &= -3 \end{aligned}\right\} \Rightarrow H(x,y) = f_{xx}f_{xy} - (f_{xy})^2 = 36x - 3$$

For $(0,0)$, $H < 0$

$\Rightarrow (0,0)$ is a saddle point.

For $(1/2, 3/4)$, $H > 0$ and $f_{xx} > 0$

$\Rightarrow (1/2, 3/4)$ is a local minimum

7. $\boxed{\text{8 points}}$ Let $f(x, y) = 8x + 2y$.

(a) Use the method of Lagrange multipliers to find the maximum and minimum values of f given the constraint $x^2 + y^2 = 17$.

Let $g(x,y) = x^2+y^2$.

We want to solve $\nabla f = \lambda \nabla g$.

$\nabla f = \langle 8, 2\rangle$ Setting $\nabla f = \lambda \nabla g$, we get

$\nabla g = \langle 2x, 2y\rangle$ $8 = \lambda 2x$ and $2 = \lambda 2y$, or $\lambda = \frac{4}{x} = \frac{1}{y}$.

Therefore, $4y = x$. Now, using the constraint and substituting $x = 4y$, we have

$(4y)^2 + y^2 = 17 \Rightarrow y^2 = 1 \Rightarrow y = 1$ or $y = -1$.

When $y = 1$, $x = 4$, and $f(4,1) = 34$. When $y=-1$, $x=-4$, and $f(-4,-1) = -34$.

So, the max of f, given $x^2+y^2=17$, is 34 and the min is -34.

(b) Using your answer from part (a), find the absolute maximum and absolute minimum of f on the closed disk $D = \{(x, y)|x^2 + y^2 \leq 17\}$. Explain your reasoning.

By the extreme value theorem, the extreme values of f either occur at a critical point of f or on the boundary of D. As f has no critical points, the extreme values occur on the boundary of D.

But we just found the extreme values of f on the boundary of D in part (a). Therefore, 34 is the absolute maximum of on D and -34 is the absolute minimum.