MATH 231: Fall 2025 Instructor: Nicholas Vlamis

Wednesday 10/15/2025 Exam 1 110 minutes

Name: Solutions

Instructions.

1. ***Read each problem carefully.*** Make sure you understand what the problem is asking.
2. Unless previously granted permission, you may only use a TI-82, TI-83, TI-84 or scientific calculator.
3. You may use a note sheet, which consists of a single sheet of 8.5" x 11" inch paper. Your note sheet is **not** allowed to contain solutions to problems or proofs of theorems. It will be collected with your exam.
4. No devices other than a writing utensil and calculator may be used.

Question	Points	Score
1	6	
2	8	
3	3	
4	2	
5	3	
6	4	
7	2	
8	7	
9	3	
10	4	
11	8	
Total:	50	

Questions

1. $\boxed{\text{6 points}}$ Consider the following system of linear equations:

$$\begin{aligned} x_1 - x_2 - x_3 &= -7 \\ 4x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_2 + 2x_3 &= 8 \end{aligned}$$

(a) Write down the augmented matrix of the linear system.

$$\begin{bmatrix} 1 & -1 & -1 & -7 \\ 4 & 4 & 2 & 0 \\ 0 & 2 & 2 & 8 \end{bmatrix}$$

(b) Write the linear system as a matrix equation.

$$\begin{bmatrix} 1 & -1 & -1 \\ 4 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 8 \end{bmatrix}$$

(c) For the next part, use the fact that the reduced row echelon form of the augmented matrix of the linear system is:

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

How many solutions does the linear system have? If it has solutions, find them all.

There is one solution: $x_1 = -3$
$x_2 = 2$
$x_3 = 2$

6. $\boxed{4 \text{ points}}$ Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation satisfying

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\\1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-1\\1\\0\end{bmatrix}$$

(a) Find the standard matrix for T.

$T = T_A$ where $A = [T(e_1) \; T(e_2)]$ and

$e_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

So $A = \begin{bmatrix}1 & 3\\1 & -1\\0 & 1\\1 & 0\end{bmatrix}$ is the standard matrix for T.

(b) Compute $T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)$.

$$T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}1 & 3\\1 & -1\\0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}5\\-3\\2\\-1\end{bmatrix}$$

7. $\boxed{2 \text{ points}}$ Let A be an 3×3 matrix satisfying

$$A\begin{bmatrix}0\\1\\3\end{bmatrix} = 0 \text{ and } A\begin{bmatrix}1\\0\\1\end{bmatrix} = 0$$

Give a 3×2 matrix B with distinct non-zero column vectors such that $AB = 0$.

Let $b_1 = \begin{bmatrix}0\\1\\3\end{bmatrix}$ and $b_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, so $Ab_1 = 0$ and $Ab_2 = 0$.

Set $B = [b_1 \; b_2] = \begin{bmatrix}0 & 1\\1 & 0\\3 & 1\end{bmatrix}$.

Then $AB = [Ab_1 \; Ab_2] = [\vec{0} \; \vec{0}] = \begin{bmatrix}0 & 0\\0 & 0\\0 & 0\end{bmatrix}$.

3. $\boxed{\text{3 points}}$ Suppose each of the following matrices is the augmented matrix for a system of linear equations. For each, write down *how many* solutions its corresponding system has. (No computations should be necessary and no work is required.)

(a) $\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

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4. $\boxed{\text{2 points}}$ Let $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -5 & 3 \\ 0 & -1 & 7 \end{bmatrix}$ and let $B = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -7 & 17 \\ 0 & -1 & 7 \end{bmatrix}$.

Find a single elementary row operation that when applied to A yields B. (Make sure you give a clear description of what the operation is.)

Multiply the 3rd row of A by 2 and add to the second row of A.

5. $\boxed{\text{3 points}}$ Suppose $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}\right) = T\left(\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right)$$

Find a nontrivial solution to $T(\mathbf{x}) = \mathbf{0}$.

$$\begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 7 \end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix} -4 \\ -5 \\ 7 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}\right) - T\left(\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right) = 0$$

$\Rightarrow X = \begin{bmatrix} -4 \\ -5 \\ 7 \end{bmatrix}$ is a solution to $T(x) = 0$

2. $\boxed{\text{8 points}}$ Let A be a 4×4 matrix such that

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & -7 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The following questions are about the linear system in the variables x_1, x_2, x_3, and x_4 that is associated to the matrix equation $A\mathbf{x} = \mathbf{0}$, where we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(a) Determine which of the variables of the linear system are basic (or leading) and which are free. (Note: A is **not** an augmented matrix.)

leading variables: x_1, x_3

Free Variables: x_2, x_4

(b) Find the (parameterized) general solution to the linear system.

$x_1 = 7s - 2t$

$x_2 = s$ for $s, t \in \mathbb{R}$

$x_3 = t$

$x_4 = t$

(c) Write the solution to $A\mathbf{x} = 0$ as the span of a collection of vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7s - 2t \\ s \\ t \\ t \end{bmatrix} = s\begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution set is $\mathrm{Span}\left\{\begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

8. $\boxed{\text{7 points}}$ Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors in $\mathbb{R}^3$. Let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$, and suppose

$$\text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ linearly independent? Explain.

No. $Ax=0$ has a free variable, and hence a nontrivial solution. But a nontrivial solution to $Ax=0$ is a nontrivial solution to $x_1v_1+x_2v_2+x_3v_3+x_4v_4=0$, implying $\{v_1, v_2, v_3, v_4\}$ is linearly dependent

(b) Is span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ all of $\mathbb{R}^3$? Explain.

Yes. A has a pivot position in every row, so $Ax=b$ has a solution for every $b \in \mathbb{R}^3$. In other words, every $b \in \mathbb{R}^3$ is in the span of $\{v_1, v_2, v_3, v_4\}$.

(c) What are the domain and codomain of the matrix transformation T_A?

A is $3\times 4 \Rightarrow$ Domain of T_A is $\mathbb{R}^4$, and the Codomain is $\mathbb{R}^3$.

(d) Is the matrix transformation T_A onto? Explain.

Yes. Given any $b \in \mathbb{R}^3$, $Ax=b$ has a solution, say $x=v$. Then $T_A(v) = Av = b$.

9. $\boxed{\text{3 points}}$ Let $\mathbf{v} \in \mathbb{R}^n$. Show that if $\mathbf{v} \cdot \mathbf{v} = 0$, then $\mathbf{v} = \mathbf{0}$.

Let's argue the contrapositive: if $\vec{v} \neq \vec{0}$, then $\vec{v}\cdot\vec{v} \neq 0$.

Write $\vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$. As $\vec{v} \neq 0$, $\exists\, i$ so that $a_i \neq 0$.

Now, $\vec{v}\cdot\vec{v} = a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_i^2 > 0$. $\square$

10. $\boxed{\text{4 points}}$ Suppose A and B are matrices such that the last column of AB is all zeroes. Give an argument showing that if B has no column of all zeroes itself, then the columns of A are linearly dependent.

Write $B = [\vec{b}_1 \ \vec{b}_2 \ \cdots \ \vec{b}_n]$.

We are told that $\vec{b}_n \neq \vec{0}$ and $A\vec{b}_n = \vec{0}$.

$\Rightarrow$ $\vec{b}_n$ is a nontrivial solution to $Ax = \vec{0}$

$\Rightarrow$ The columns of A are linearly dependent.

11. $\boxed{\text{8 points}}$ Decide whether each of the following statements is TRUE or FALSE (no explanation required).

(a) A matrix equation of the form $A\mathbf{x} = \mathbf{0}$ always has at least one solution.

True

(b) There exists a linear system with exactly three solutions.

False

(c) Every linear system with the same number of unknowns and equations has exactly one solution.

False

(d) If A and B are square matrices of the same size, then $AB = BA$.

False

(e) If $T\colon \mathbb{R}^p \to \mathbb{R}^q$ is a linear transformation, then the standard matrix for T has size $q \times p$.

True

(f) Any set containing exactly one vector is linearly independent.

False

(g) The matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if every column of A is a pivot column.

False

(h) If the rightmost column of an augmented matrix is a pivot column, then the associated linear system is inconsistent.

True