MATH 231: Fall 2025 **Instructor: Nicholas Vlamis**

Wednesday 11/19/2025 Exam 2 110 minutes

Name: Solutions

Instructions.

1. ***Read each problem carefully.*** Make sure you understand what the problem is asking.
2. Unless previously granted permission, you may only use a TI-82, TI-83, TI-84 or scientific calculator.
3. You may use a note sheet, which consists of a single sheet of 8.5" x 11" inch paper. Your note sheet is **not** allowed to contain solutions to problems or proofs of theorems. It will be collected with your exam.
4. No devices other than a writing utensil and calculator may be used.

Question	Points	Score
1	5	
2	6	
3	5	
4	6	
5	10	
6	10	
7	4	
8	4	
Total:	50	

Theorem 1 (Invertible Matrix Theorem)**.** Let A be an $n \times n$ matrix. Then the following statements are equivalent.

a. A is an invertible matrix.

b. A is row equivalent to I_n.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I_n$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\mathrm{col}(A) = \mathbb{R}^n$.

o. $\mathrm{rank}(A) = n$.

p. $\mathrm{nullity}(A) = 0$.

q. $\mathrm{null}(A) = \{\mathbf{0}\}$.

r. $\det(A) \neq 0$.

s. 0 is not an eigenvalue of A.

Questions

1. $\boxed{\text{5 points}}$ Let A be a 3×3 matrix with $\det(A) = -3$.

   (a) Find $\det(2A)$.

   $\det(2A) = 2^3 \det(A) = 8(-3) = -24$

   (b) Find $\det(A^{-1})$.

   $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-3}$

   (c) Find $\det(A^2)$.

   $\det(A^2) = \det(AA) = \det(A)\det(A) = (-3)^2 = 9$

   (d) Suppose B is a matrix obtained from A by multiplying the first row of A by 2, the second row by -3, and switching the last two rows of A. Find $\det(B)$.

   $\det(B) = (2)(-3)(-1)\det(A) = 6(-3) = -18$

   (e) Find $\det(AB)$.

   $\det(AB) = \det(A)\det(B) = (-3)(-18) = 54$

2. $\boxed{\text{6 points}}$ Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & -2 \\ 5 & 10 & 2 & -5 \end{bmatrix}$ whose reduced row echelon form is as follows

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Is the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \in \mathbb{R}^4$ in the null space of A? How do you know?

By definition, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ is in the null space of A if $A\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \vec{0}$.

Let's check: $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & -2 \\ 5 & 10 & 2 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 4-4 \\ 10-10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ is in the null space of A.

(b) Find a basis for col(A).

The pivot columns of A give a basis for col(A),

so $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for col(A).

(c) Find the rank and nullity of A.

The rank nullity theorem in this case says

$\text{rank}(A) + \text{nullity}(A) = 4$.

By part (b), $\text{rank}(A) = 2$ and so

$\text{nullity}(A) = 4 - 2 = 2$

3. $\boxed{\text{5 points}}$ Let A be an invertible matrix whose inverse is given by

$$A^{-1} = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 3 & 1 & -5 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) Let $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. Find a solution to $A\mathbf{x} = \mathbf{b}$.

If $Ax=b$, then $A^{-1}(Ax) = A^{-1}b$

$$\Rightarrow x = A^{-1}b = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 3 & 1 & -5 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ 0 \\ 5 \end{bmatrix}$$

(b) Using only the definition of an invertible matrix, provide an argument for the fact that $A\mathbf{x} = \mathbf{b}$ has exactly one solution, that is, show that if $\mathbf{u}$ and $\mathbf{v}$ are both solutions to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{v} = \mathbf{u}$. (To be clear, you cannot use the Invertible Matrix Theorem in your argument.)

If $Au = Av$, then $A^{-1}(Au) = A^{-1}(Av)$.

$\Rightarrow (A^{-1}A)v = (A^{-1}A)u$

$\Rightarrow Iv = Iu$

$\Rightarrow v = u$.

4. $\boxed{\text{6 points}}$ Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ be vectors in $\mathbb{R}^2$, and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) Provide an argument showing that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

Consider the matrix $B = [v_1 \ v_2] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$.

The invertible matrix theorem tells us that if B is invertible, then $\mathcal{B}$ is linearly independent and $\mathcal{B}$ spans $\mathbb{R}^2$; in other words that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

To check the invertibility of B, we can just check that its determinant is nonzero.

$$\det B = -5 \neq 0 \Rightarrow \mathcal{B} \text{ is a basis for } \mathbb{R}^2.$$

(b) Find the $\mathcal{B}$-coordinates of $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, that is, compute $[\mathbf{v}]_{\mathcal{B}}$.

By definition, $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where $v = c_1 v_1 + c_2 v_2$.

This says that $B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = v$

$$\Rightarrow [v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -7 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

(c) Suppose A is a 2×2 matrix such that $A\mathbf{v}_1 = 3\mathbf{v}_1$ and $A\mathbf{v}_2 = -2\mathbf{v}_2$. Compute $[A\mathbf{v}]_{\mathcal{B}}$. (You will have to use your answer from (b).)

$$\begin{aligned} Av = A\left(\tfrac{7}{5} v_1 + \tfrac{1}{5} v_2\right) &= (7/5)\, Av_1 + (1/5)\, Av_2 \\ &= 7/5\,(3v_1) + (1/5)(-2v_2) \\ &= \left(\tfrac{21}{5}\right) v_1 + \left(\tfrac{-2}{5}\right) v_2 \end{aligned}$$

$$\Rightarrow [Av]_{\mathcal{B}} = \begin{bmatrix} 21/5 \\ -2/5 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 21 \\ -2 \end{bmatrix}$$

5. $\boxed{\text{10 points}}$ Let $A = \begin{bmatrix} 1 & 6 & -6 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

(a) Is $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ an eigenvector of A? Explain how you know.

By definition, v is an eigenvector of A if $Av = \lambda v$ for some constant λ. Let's check:

$$Av = \begin{bmatrix} 1 & 6 & -6 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix}.$$

Setting $Av = \lambda v$, we get the following equations from the first two coordinates: $1 = \lambda(-5)$ and $0 = 2\lambda$, which are inconsistent $\Rightarrow v$ is not an e. vector

(b) Find the characteristic polynomial of A. Show your work. (You should get the polynomial $(\lambda-1)^2(\lambda+2)$.)

The characteristic polynomial is:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -6 & 6 \\ 0 & \lambda+1 & -2 \\ 0 & -1 & \lambda \end{vmatrix}$$

$$= (\lambda-1)[(\lambda+1)\lambda - 2]$$

$$= (\lambda-1)[\lambda^2+\lambda-2] = (\lambda-1)(\lambda-1)(\lambda+2) = (\lambda-1)^2(\lambda+2)$$

(c) What are the eigenvalues of A?

The eigenvalues of A correspond to the roots of its characteristic polynomial $\Rightarrow \lambda = 1$ and $\lambda = -2$ are the eigenvalues of A

(Problem continued on next page.)

(d) Find a basis for each eigenspace. (Feel free to work on the back of the sheet.)

$\lambda = 1$:

$E_1 = \text{null}(I - A)$

$= \text{null}\left(\begin{bmatrix} 0 & -6 & 6 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}\right)$

$\text{rref}(I-A) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ Solution to $(I-A)x = \vec{0}$ is $x_1 = t$, $x_2 = s$, $x_3 = s$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for E_1

$\lambda = -2$:

$E_{-2} = \text{null}(-2I - A)$

$= \text{null}\left(\begin{bmatrix} -3 & -6 & 6 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}\right)$

G-J elimination:

$\begin{bmatrix} -3 & -6 & 6 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & -6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$= \text{rref}(-2I-A)$

$\Rightarrow$ Solution to $(-2I-A)x = 0$ is $x_1 = 6t$, $x_2 = -2t$, $x_3 = t$

$\Rightarrow \left\{ \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for E_{-2}

(e) What is the dimension of each of the eigenspaces of A?

$\dim E_1 = 2$

$\dim E_{-2} = 1$

6. $\boxed{\text{10 points}}$ Answer the following questions. No explanation is necessary.

(a) If A is 3×2 matrix and the matrix product AB is a 3×4 matrix, what is the size of B?

B is 2×4

(b) Write down the 3×3 elementary matrix corresponding to the row operation of switching the first and third row.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) Suppose A is a 5×7 matrix. What is the value of n for which the column space of A a subspace of $\mathbb{R}^n$?

col(A) is a subspace of $\mathbb{R}^5$

(d) Suppose A is a 5×7 matrix. What is the value of n for which the null space of A a subspace of $\mathbb{R}^n$?

null(A) is a subspace of $\mathbb{R}^7$

(e) Suppose A is a square matrix that is **not** invertible. What is $\det(A)$?

0

(f) If A is a 9×7 matrix and the rank of A is 3, then what is the nullity of A?

4

(g) What is the rank of an invertible 6×6 matrix?

6

(h) List all the possible values for the nullity of a 3×5 matrix.

rank is either 0, 1, 2, 3 so nullity is either 5, 4, 3, 2

(i) Let A be a 8×6 matrix. If A has four pivot columns, what is the size of a basis for the solution set to $A\mathbf{x} = 0$?

2

(j) Given nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, what is the dimension the subspace span$\{\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}\}$?

2

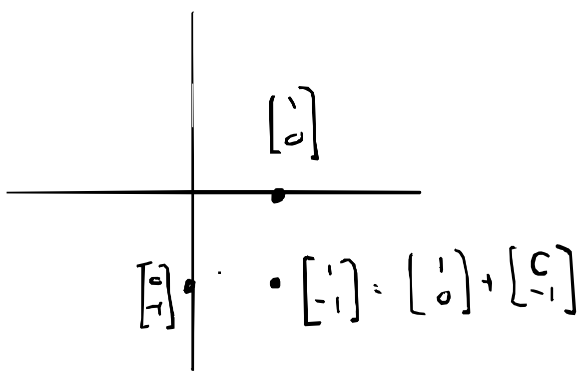
7. $\boxed{\text{4 points}}$ Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : xy \geq 0 \right\}$, so that W is the union of the first and third quadrants in the xy-plane. Determine whether W is a subspace of $\mathbb{R}^2$ or not. Justify your answer.

No, W is not a subspace.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are in W,

but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not,

so W cannot be a subspace.



8. $\boxed{\text{4 points}}$ Use the properties of the determinant to show that if A and B are square matrices such that AB is invertible, then A and B are both invertible.

We know $\det(AB) \neq 0$, as AB is invertible.

Now, $\det(AB) = \det(A)\det(B)$

$\Rightarrow \det(A)\det(B) \neq 0$

$\Rightarrow \det(A) \neq 0$ and $\det(B) \neq 0$

$\Rightarrow$ A and B are both invertible.