# Homework 3

MATH 301/601

Test #3 is on Wednesday, March 12

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**Instructions.** *Read the Homework Guide to make sure you understand how to successfully complete the assignment.*

**Exercise 1.** Prove that

$$G = \{a + b\sqrt{2} : a, b \in \mathbb{Q} \text{ and } a \text{ and } b \text{ are not both zero}\}$$

is a subgroup of $\mathbb{R}^\times$ under the group operation of multiplication.

***Exercise 2.** Let $H$ and $K$ be subgroup of a group $G$.

(a) Prove that $H \cap K$ is a subgroup of $G$.

(b) Prove or disprove: $H \cup K$ is a subgroup of $G$.

(c) Prove that if $G$ is abelian, then $HK = \{hk : h \in H, k \in K\}$ is a subgroup of $G$.

**Exercise 3.** Let $G$ be a group. The *center* of $G$ is the set

$$Z(G) = \{a \in G : ga = ag \text{ for all } g \in G\}.$$

Prove that $Z(G)$ is a subgroup of $G$.

**Exercise 4.** (a) Compute the center of $\mathrm{GL}(n, \mathbb{R})$. Hint: Use the following test matrices: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) Compute the center of $\mathrm{SL}(n, \mathbb{R})$.

***Exercise 5.** Let $H$ be a subgroup of a group $G$. Define the relation $\sim$ on $G$ by $a \sim b$ if $b^{-1}a \in H$. Prove that $\sim$ is an equivalence relation on $G$.

****Exercise 6.** Suppose $H$ is a nonempty finite subset of a group $G$ and that $H$ is closed under multiplication (that is, $ab \in H$ for all $a, b \in H$). Prove that $H$ is a subgroup of $G$.

**Exercise 7.** Let $G$ be a group, and let $a \in G$ have finite order.

(a) Prove that the set $\{k \in \mathbb{N} : a^k = e\}$ is not empty.

(b) Let $n \in \mathbb{N}$ be the least element of the set $\{k \in \mathbb{N} : a^k = e\}$ (which exists by part (a) and the well-ordering principle). Prove that if $i, j \in \{0, \ldots, n-1\}$ such that $a^i = a^j$, then $i = j$.

(c) Prove that $\langle a \rangle = \{a^0, a^1, \ldots, a^{n-1}\}$.

(Note: Parts (b) and (c) together show that $|a| = n$.)

**Exercise 8.** Determine if $\mathbb{Q}$ is cyclic. Justify your answer.

***Exercise 9.** Let $A \in \mathrm{GL}(2, \mathbb{R})$.

(a) Prove that if $A$ has finite order, then $\det(A) = \pm 1$.

(b) Prove that if $\det(A) = 1$ and $|\mathrm{tr}(A)| > 2$, then $A$ has a (real) eigenvalue $\lambda$ such that $|\lambda| \neq 1$, where $\mathrm{tr}(A)$ denotes the *trace* of $A$.

(c) Let $A$ be as in part (b), so $\det(A) = 1$ and $|\mathrm{tr}(A)| > 2$. Use the existence of an eigenvalue (and hence an eigenvector) to prove that if $\det(A) = 1$, then $A$ has infinite order.

**Exercise 10.** Let $G$ be a group

(a) Let $a, g \in G$. Prove that $|a| = |gag^{-1}|$.

(b) Let $a, b \in G$. Prove that $|ab| = |ba|$. (Hint: use part (a).)

**Exercise 11.** Let $p$ be a prime number. Prove that $\mathbb{Z}_p$ has exactly two subgroups, namely the trivial subgroup and itself.

***Exercise 12.** Suppose $G$ is a nontrivial group in which the only two subgroups of $G$ are itself and the trivial subgroup.

(a) Prove that $G$ is cyclic.

(b) Using part (a), prove that $G$ is a finite group of prime order.

**Exercise 13.** Let $p$ and $q$ be distinct prime numbers.

(a) How many generators does $\mathbb{Z}_{pq}$ have? Justify your answer.

(b) Let $r \in \mathbb{N}$. How many generators does $\mathbb{Z}_{p^r}$ have? Justify your answer.

***Exercise 14.** Let $a$ be an element of a group. For $n, m \in \mathbb{Z}$, find a generator for the group $\langle a^m \rangle \cap \langle a^n \rangle$. Justify your answer.

**Exercise 15.** Let $a$ and $b$ be elements in a group with relatively prime orders. Prove that $\langle a \rangle \cap \langle b \rangle$ is the trivial subgroup.

****Exercise 16.** Let $p, q \in \mathbb{N}$ be relatively prime, and let $G$ be an abelian group of order $pq$. Prove that if $G$ contains elements of order $p$ and $q$, then $G$ is cyclic.

**Exercise 17.** Let $G$ be an abelian group. Show that the elements of finite order in $G$ form a subgroup (known as the **torsion subgroup**).