Homework 4

MATH 301/601

Test #4 is on Wednesday, April 2

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment.*

Exercise 1. In Section 5.4 of the textbook, complete exercises 1, 2(a,b,c,d), and 4.

Exercise 2. Determine if each of the following subsets of S_4 is a subgroup or not.

(a) $\{\sigma \in S_4 : \sigma(1) = 3\}$

(b) $\{\sigma \in S_4 : \sigma(2) = 2\}$

(c) $\{\sigma \in S_4 : \sigma(\{2,3\}) = \{2,3\}\}$

Exercise 3. Prove that the order of S_n is $n!$ (recall that $n! = n(n-1)(n-2)\cdots 2\cdot 1$).

***Exercise 4.** Choose examples of 2-, 3-, 4-, and 5-cycles, and compute the cyclic subgroups they generate. Use these examples to conjecture the order of a k-cycle. Prove your conjecture.

***Exercise 5.** A 2-cycle is called a *transposition*. Prove that a k-cycle can be expressed as a product of $k-1$ transpositions.

Exercise 6. For $n \geq 3$, prove that the center of S_n is trivial (that is, it only contains the identity element).

Exercise 7. Let σ be a k-cycle. Prove that k is odd if and only if σ^2 is a cycle.

****Exercise 8.** ***Oops***: This is the same as Exercise 16, but Exercise 16 is more scaffolded, so just do that one. Prove that any two k-cycles in S_n are *conjugate*, that is, if $\sigma, \tau \in S_n$ are k-cycles, then there exists $\mu \in S_n$ such that $\mu\sigma\mu^{-1} = \tau$.

Exercise 9. What are the possible cycle structures of elements in A_5? What about A_6?

***Exercise 10.** Prove that in A_n with $n \geq 3$, any permutation is a product of cycles of length 3.

****Exercise 11.** Label the vertices of a *tetradhedron* by 1, 2, 3, and 4 (see Figure 1). For each of the permutations $(1\,2\,3)$ and $(12)(34)$ in A_4, describe a rigid motion of the tetrahedron that induces the permutation.

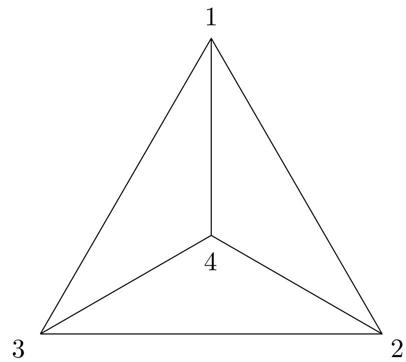


Figure 1: A tetrahedron with vertices labelled.

Exercise 12. A series of questions dealing with cycles decompositions and orders.

(a) Find all possible orders of elements in S_7.

(b) Find all possible orders of elements in A_7.

(c) Show that A_{10} contains an element of order 15.

(d) Does A_8 contain an element of order 26?

(e) What are the possible cycle decomposition structures of elements in A_5? What about A_6?

***Exercise 13.** The goal of this exercise is to deduce the order of the alternating group A_n for $n \in \mathbb{N}$. Throughout the exercise, let B_n be the subset of S_n consisting of odd permutations (recall that A_n is the subgroup of S_n consisting of even permutations).

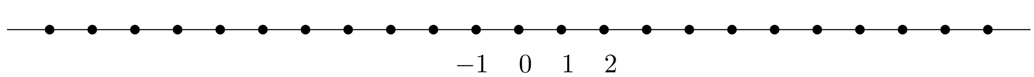
(a) Prove that $A_n \cap B_n = \varnothing$.

(b) Fix $\tau \in B_n$, and define $f\colon A_n \to B_n$ by $f(\sigma) = \tau\sigma$. Prove that f is a bijection.

(c) Use the previous parts, together with the facts that $S_n = A_n \cup B_n$ and $|S_n| = n!$, to deducde that $|A_n| = \frac{n!}{2}$.

***Exercise 14.** Let $n \in \mathbb{N}$. Prove that in A_n with $n \geq 3$, any permutation is a product of 3-cycles.

***Exercise 15.** This exercises provides a more formal approach to a problem you worked on on an earlier homework. Let $\Gamma = (V, E)$ be the graph with $V = \mathbb{Z}$ and $(m, n) \in \mathbb{Z}$ if and only if $|m - n| = 1$. So, Γ is just the number line (a portion of which is drawn here):



The *infinite dihedral group*, denoted D_∞, is the automorphism group of the graph Γ. Let $\tau, \rho \in D_\infty$ be given by $\tau(n) = n + 1$ and $\rho(n) = -n$ for $n \in \mathbb{Z}$.

(a) For $k \in \mathbb{Z}$, write down a formula for τ^k.

(b) Prove that if $f \in D_\infty$ such that $f(0) = 0$ and $f(1) = 1$, then f is the identity. (Hint: Let's first focus on the natural numbers. Use strong induction: Let $k \in \mathbb{N} \smallsetminus \{1\}$. Suppose that $f(j) = j$ for all $0 \leq j < k$ and prove that $f(k) = k$. A similar argument works for the negative integers.)

(c) Prove that every element of D_∞ can be written as either τ^k or $\tau^k \rho$ for some $k \in \mathbb{Z}$. (Hint: let $f \in D_\infty$. Use a power of τ to get $f(0)$ back to 0, and then use ρ to get 1 back to itself if necessary.)

****Exercise 16.** Let $\tau = (a_1\, a_2\, \ldots, a_k)$ be a k-cycle.

(a) Prove that if σ is any permutation, then

$$\sigma\tau\sigma^{-1} = (\sigma(a_1)\,\sigma(a_2)\,\ldots\sigma(a_k))$$

is a k-cycle.

(b) Let μ be a k-cycle. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.