Homework 5

MATH 301/601

Test #5 is on Wednesday, April 23

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment.*

Exercise 1. Let G be a group. Define the relation $\sim$ on G as follows: $a \sim b$ if and only if b is *conjugate* to a (that is, there exists $g \in G$ such that $b = gag^{-1}$). Prove that $\sim$ is an equivalence relation.

Exercise 2. Prove that the 3-cycles $(1\,2\,3)$ and $(1\,3\,2)$ are **not** conjugate in A_4.

Exercise 3. In http://abstract.ups.edu/aata/cosets-exercises.html of the textbook, complete Exercises 1, 2, 3, 4, 5(a)-(f), and 6.

***Exercise 4.** Let H be a subgroup of a group G and let $g_1, g_2 \in G$.

(a) Prove that if $g_1H = g_2H$, then $g_2^{-1}g_1 \in H$.

(b) Prove that if $g_1H \subset g_2H$, then $g_1H = g_2H$.

***Exercise 5.** Let H be a subgroup of G. Prove that if $ghg^{-1} \in H$ for all $g \in G$ and for all $h \in H$, then $aH = Ha$ for all $a \in G$.

Exercise 6. Let G be a group and H a subgroup of G. Suppose that $[G : H] = 2$. Prove that if $a, b \in G$ are not in H, then $ab \in H$.

***Exercise 7.** Let G be a group and let H be a subgroup of G. Prove that if $[G : H] = 2$, then $aH = Ha$ for all $a \in G$.

***Exercise 8.** Let $n \in \mathbb{N}$. Use Fermat's Little Theorem to show that if $p = 4n + 3$ is prime, then there is no solution to the equation $x^2 \equiv -1 \pmod{p}$.

Exercise 9. Let $p \in \mathbb{N}$ be prime. How many subgroups does $\mathbb{Z}_{2p}$ have? Prove it.

Exercise 10. Let $\varphi : G \to H$ be an isomorphism of groups.

(a) Prove that $\varphi(e_G) = e_H$. (Hint: usse the fact that $e_G e_G = e_G$.)

(b) Prove that $\varphi(g)^{-1} = \varphi(g^{-1})$ for all $g \in G$.

(c) Prove that $\varphi(g^n) = \varphi(g)^n$ for all $g \in G$ and for all $n \in \mathbb{Z}$.

(d) Prove that $\varphi^{-1} : H \to G$ is an isomorphism.

***Exercise 11.** Prove that $\mathbb{C}^\times$ is isomorphic to the subgroup of $\mathrm{GL}(2, \mathbb{R})$ consisting of matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Exercise 12. Prove that S_4 is not isomorphic to D_4.

Definition 1. An *automorphism* of a group G is an isomorphism $G \to G$.

***Exercise 13.** Let G be a finite abelian group of order n. Suppose $m \in \mathbb{N}$ is relatively prime to n. Prove that $\varphi : G \to G$ given by $\varphi(g) = g^m$ is an automorphism of G. (This says that every element of G has an m^{th}-root.)

Exercise 14. Let G be a group. Prove that the set of automorphisms of G, denoted $\mathrm{Aut}(G)$, is a group with respect to function composition (this group is called the *automorphism group* of G.).

***Exercise 15.** (a) Let G be a cyclic group, and let $\varphi, \psi \in \mathrm{Aut}(G)$. Prove that if $a \in G$ is a generator of G and $\varphi(a) = \psi(a)$, then $\varphi = \psi$.

(b) Use part (a) to compute $\mathrm{Aut}(\mathbb{Z})$.

****Exercise 16.** Use part (a) of the previous exercise, together with Exercise 13, to show that $\mathrm{Aut}(\mathbb{Z}_n)$ is isomorphic to $U(n)$.

Double-star problem set up[1]

Definition 2. Let G be a group, and let X be a set. A *group action* of G on X is a function $\phi\colon G \times X \to X$ satisfying:

(i) $\phi(e, x) = x$ for all $x \in X$, and

(ii) $\phi(gh, x) = \phi(g, \phi(h, x))$ for all $g, h \in G$ and for all $x \in X$.

Usually the group action is clear from context and we simply write gx or $g.x$ instead of $\phi(g, x)$. In this notation, (i) says $e.x = x$ for all $x \in X$, and (ii) says $(gh).x = g.(h.x)$. Again suppressing the function ϕ, we generally write $G \curvearrowright X$ to denote the fact that the group G is acting on the set X.

In the "real world", we generally think about a group by the way it acts on some set. For example, we think about the dihedral groups via their action on regular polygons, and we think of matrix groups via their action on vector spaces.

[1]See Section 14.1

Definition 3. Let $G \curvearrowright X$. Given $x \in X$, the *orbit* of x, denoted $\mathcal{O}_x$, is the subset of X given by

$$\mathcal{O}_x = \{g.x : g \in G\}$$

and the *stabilizer* of x, denoted $\mathrm{Stab}_G(x)$ is the subgroup[2] of G given by

$$\mathrm{Stab}_G(x) = \{g \in G : g.x = x\}.$$

The goal of the next exercise is to prove the following:

Theorem 4 (Orbit–Stabilizer Theorem)**.** *Let G be a group acting on a set X. If $x \in X$, then $|G| = |\mathcal{O}_x| \cdot |\mathrm{Stab}_G(x)|$.*

The orbit–stabilizer theorem should be viewed as a generalization of Lagrange's theorem (which we will use to prove the orbit–stabilizer theorem). Indeed, let H be a subgroup of G, and let $\mathcal{L}_H$ be the left cosets of H. Then G acts on $\mathcal{L}_H$ by $g.(aH) = (ga)H$, with $\mathrm{Stab}_G(H) = H$ and $\mathcal{O}_H = \mathcal{L}_H$.

****Exercise 17.** Let G be a group acting on a set X. Let $x \in X$.

(a) Let $g, h \in G$. Prove that $gx = hx$ if and only if $h^{-1}g \in \mathrm{Stab}_G(x)$.

(b) Let $\mathcal{L}$ be the set of left cosets of $\mathrm{Stab}_G(x)$ in G. Let $\psi \colon \mathcal{L} \to \mathcal{O}_x$ be given by $\psi(g\mathrm{Stab}_G(x)) = gx$.

  (i) Prove that ψ is a well-defined, that is, prove that if $g\mathrm{Stab}_G(x) = h\mathrm{Stab}_G(x)$, then $g.x = h.x$.

  (ii) Prove that ψ is bijective.

The previous part implies that $|\mathcal{O}_x| = [G : \mathrm{Stab}_G(x)]$. Lagrange's theorem tells us that $|G| = |\mathcal{O}_x| \cdot |\mathrm{Stab}_G(x)|$, yielding the orbit-stabilizer theorem.

****Exercise 18.** Let G be a finite group. Use the orbit–stabilizer theorem to show that, for $a \in G$, the cardinality of the set $\{gag^{-1} : g \in G\}$ (that is, the conjugacy class of a) divides $|G|$.

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[2] You should convince yourself that this is indeed a subgroup.