Homework 6

MATH 301/601
Test #6 is on Wednesday, May 7

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment.*

Exercise 1. Find five non-isomorphic groups of order 8. Justify why no two of them are isomorphic. (You will need to learn about the *quaternion group*, see Example 3.15 in the book.)

Exercise 2. Let G be a group of order 20. If G has subgroups H and K of orders 4 and 5, respectively, such that $hk = kh$ for all $h \in H$ and $k \in K$, prove that G is the internal direct product of H and K.

***Exercise 3.** (a) Prove or disprove: there is a noncyclic abelian group of order 51. (Do not use the classification of finite abelian groups.)

(b) Prove or disprove: there is a noncyclic abelian group of order 52.

Exercise 4. Prove that D_4 cannot be the internal direct product of two of its proper subgroups.

***Exercise 5.** Recall that we can express each element of D_6 as a product of r and s, where $r, s \in D_6$ satisfy $|r| = 6$, $|s| = 2$, and $sr = r^{-1}s$; in particular, we have

$$D_6 = \{\text{id}, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}.$$

Let $H = \langle r^3 \rangle$ and let $K = \{\text{id}, r^2, r^4, s, sr^2, sr^4\}$.

(a) Prove that D_6 is the internal direct product of H and K.

(b) Show that K is isomorphic to S_3.

(c) Deduce that $D_6 \cong S_3 \times \mathbb{Z}_2$.

***Exercise 6.** The goal of this exercise is to prove the folowing statement:

Every group of order four is isomoprhic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Let G be a group of order 4. If G has an element of order 4, then it must be cyclic, and hence isomorphic to $\mathbb{Z}_4$. So, assume that G does not have an element of order 4; the goal is now to prove that G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(a) Prove that every non-identity element of G has order two.

(b) Let $g, h \in G$ be distinct elements. Let $K = \langle g \rangle$ and let $H = \langle h \rangle$. Prove that G is the internal direct product of H and K.

(c) Use the above to prove that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise 7. Let G be a group, and let H and K be subgroups of G such that G is the internal direct product of H and K. Define the function $\varphi\colon H \times K \to G$ by $\varphi(h, k) = hk$. Prove that φ is an isomorphism.

Exercise 8. For each of the following groups, find all their subgroups, determine which are normal, and classify the corresponding factor groups up to isomorphism.

(a) the dihedral group D_4.

(b) the quaternion group Q_8.

***Exercise 9.** Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$, that is, matrices of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Let U consist of the matrices of the form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

where $x \in \mathbb{R}$.

(a) Show that U is a subgroup of T.

(b) Prove that U is abelian.

(c) Prove that U is normal in T.

(d) Show that T/U is abelian.

(e) Is T normal in $\mathrm{GL}(2, \mathbb{R})$?

Exercise 10. Prove that the intersection of two normal subgroups is a normal subgroup.

***Exercise 11.** If a group G has exactly one subgroup H of order k, prove that H is normal in G.

****Exercise 12.** Let G be a group. Given $a, b \in G$, define $[a, b] = aba^{-1}b^{-1}$, the group element $[a, b]$ is called the *commutator* of a and b. The *commutator subgroup* of G, denoted $[G, G]$, is the subgroup generated by the set of commutators, ie, it is the subgroup consisting of products of commutators.

(a) Prove that $[G, G]$ is a normal subgroup of G.

(b) Prove that $G/[G, G']$ is abelian.

(c) Let N be a normal subgroup of G. Prove that G/N is abelian if and only if $[G, G] \subset N$.

(The group $G/[G, G]$ is called the *abelianization* of G.)

Exercise 13. Let $\varphi\colon G_1 \to G_2$ be a homomorphism.

(a) Prove that $\varphi(G_1) = \{\varphi(g) : g \in G_1\}$ is a subgroup of G_2.

(b) Prove that if G_1 is abelian, then $\varphi(G_1)$ is abelian.

(c) Prove that if G_1 is cyclic, then $\varphi(G_1)$ is cyclic.

(d) Prove that if H is a subgroup of G_2, then $\varphi^{-1}(H) = \{g \in G_1 : \varphi(g) \in H\}$ is a subgroup of G_1.

Exercise 14. Let G be a cyclic group and let a be a generator of G. If $\varphi_1, \varphi_2\colon G \to H$ are homomorphisms such that $\varphi_1(a) = \varphi_2(a)$, prove that $\varphi_1 = \varphi_2$.

Exercise 15. (a) Find all homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_6$.

(b) Explain why there is no homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_4$ that sends $\bar{1}$ in $\mathbb{Z}_6$ to $\bar{1}$ in $\mathbb{Z}_4$.

(c) Find all the homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$.

Definition. The *kernel* of a homomorphism $\varphi\colon G \to H$, denoted $\ker\varphi$, is defined by $\ker\varphi = \{g \in G : \varphi(g) = e_H\}$.

***Exercise 16.** Let $\varphi\colon G \to H$ be a homomorphism. Prove that $\ker\varphi$ is a normal subgroup of G.

***Exercise 17.** Prove that homomorphism $\varphi\colon G \to H$ is injective if and only if $\ker\varphi = \{e_G\}$.