Test 1

Math 301/601

NAME: **Solutions**

Instructions. Read each problem carefully. Proofs can be informal: use of logical symbols and incomplete sentences **are** permitted. However, your proofs must be logically clear and correct. No devices other than a writing utensil may be used.

Problem 1. (a) Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Give the precise definition of what it means to say b *divides* a.

(b) Let $a, b \in \mathbb{Z} \setminus \{0\}$. Prove that if there exists $s, t \in \mathbb{Z}$ such that $as + bt = 1$, then $\gcd(a, b) = 1$.

Solution. (a) We say b divides a if there exists $k \in \mathbb{Z}$ such that $a = bk$.

(b) Let $d = \gcd(a, b)$. As $d \mid a$ and $d \mid b$, there exists $k, \ell \in \mathbb{Z}$ such that $a = dk$ and $b = d\ell$. This allows us to write $1 = as + bt = dks + d\ell t = d(ks + \ell t)$. Therefore, by the definition of divides, $d \mid 1$. As $d \in \mathbb{N}$ and 1 has only one factor, namely itself, we can conclude that $d = 1$. $\square$

Problem 2. Let a and b be nonzero integers. Prove that if $k \in \mathbb{Z}$ is a common multiple of a and b, then $\operatorname{lcm}(a, b)$ divides k. (Hint: divide k by $\operatorname{lcm}(a, b)$ using the division algorithm.)

Solution. Let $\ell = \operatorname{lcm}(a, b)$. By the division algorithm, there exists $q, r \in \mathbb{Z}$ such that $k = \ell q + r$ and $0 \leq r < \ell$. Rewriting, we have $r = k - \ell q$. As $a \mid k$ and $a \mid \ell$, we have that $a \mid k - \ell q$ (the argument is the same as we made above). Therefore, $a \mid r$. Similarly, $b \mid r$. This establishes r as a common multiple of a and b. As $r < \ell$ and ℓ is the smallest positive common multiple of a and b, we see that $r \leq 0$. But $r \geq 0$ and hence $r = 0$. $\square$