Test 4 Math 301/601

NAME: Solutions

Problem 1. A 2-cycle is called a *transposition.* Prove that a k-cycle can be expressed as product of $k-1$ transpositions.

Solution. We will argue using induction. The statement is clearly true for $k=2$, as a 2-cycle is itself a transposition. Now fix $k \in \mathbb{N}$ and assume that every k-cycle is a product of $k-1$ transpositions. We will show that every that $(k+1)$-cycle is a product of k transpositions, at which point the proof will be complete by the principle of induction.

Let $\sigma = (a_1\, a_2\, \cdots\, a_{k+1})$ be a $(k+1)$-cycle. We claim that $\sigma = (a_1\, a_2\, \cdots\, a_k)(a_k\, a_{k+1})$. Indeed, this can be verified in several cases, first for elements of the form a_i with $i < k$, then for a_k, then a_{k+1}, and finally for $x \neq a_i$; I leave it to you. By the inductive hypothesis, $(a_1\, a_2\, \cdots\, a_k)$ is product of $k-1$ transpositions, and so we see that σ is a product of k transpositions. □

Problem 2. Let $n \in \mathbb{N}$ such that $n \geq 3$. Prove that every permutation in A_n can be expressed as a product of 3-cycles.

Solution. We start by claiming that the product of two tranpositions can be expressed as a product of 3-cycles. We do so by cases. Let τ and ρ be two transpositions. If $\rho = \tau$, then $\tau\rho$ is the identity, which we can write as $(1\,2\,3)^3$, a product of 3-cycles. Now, if $\rho \neq \tau$, either τ and ρ are disjoint or not. If they are not disjoint, then we have three distinct numbers a, b, and c such that $\tau = (a\,b)$ and $\rho = (a\,c)$. In this case,

$$\tau\rho = (a\,b)(a\,c) = (a\,c\,b).$$

Finally, if τ and ρ are disjoint, then there exists four pairwise-distinct numbers a, b, c, and d such that $\tau = (a\,b)$ and $\rho = (c\,d)$. In this case,

$$\tau\rho = (a\,b)(c\,d) = (a\,b\,c)(b\,c\,d).$$

This finishes the proof of the claim.

Now, let $\sigma \in A_n$. Then σ is an even permutation and so there exists transpositions $\tau_1, \ldots, \tau_{2k}$ such that $\sigma = \tau_1\tau_2 \cdots \tau_{2k}$. For $i \in \{1, \ldots, k\}$, let $\mu_i = \tau_{2i-1}\tau_{2i}$. Then, by the above claim, μ_i is a product of 3-cycles. To finish, observe that $\sigma = \mu_1\mu_2 \cdots \mu_k$, and hence σ is a product of 3-cycles. □