Test 5

Math 301/601

NAME: Solutions

Problem 1. Let H be a subgroup of a group G and let $g_1, g_2 \in G$. Prove that if $g_1H \subset g_2H$, then $g_1H = g_2H$.

Solution. As we already know $g_1H \subset g_2H$, in order to show equality, we must show that $g_2H \subset g_1H$. To begin, as $g_1 \in g_1H \subset g_2H$, there exists $h \in H$ such that $g_1 = g_2h$, and hence $g_2 = g_1h^{-1}$. Now, an arbitrary element of g_2H has the form g_2h' with $h' \in H$. So, let $h' \in H$; we must show that $g_2h' \in g_1H$. Subsituting, we have $g_2h' = (g_1h^{-1})h' = g_1(h^{-1}h')$. As H is a subgroup and $h, h' \in H$, we know that $h^{-1}h' \in H$, and hence $g_2h' \in g_1H$. $\square$

Problem 2. Let G be a finite abelian group of order n. Suppose $m \in \mathbb{N}$ is relatively prime to n. Prove that the function $\varphi\colon G \to G$ given by $\varphi(g) = g^m$ is injective. (On the homework, you were asked to prove that φ is an isomorphism, so here I am only asking a portion of the question.)

Solution. Suppose $\varphi(a) = \varphi(b)$, which implies that $a^m = b^m$. Hence, $a^mb^{-m} = e$. As G is abelian, we can write $a^mb^{-m} = (ab^{-1})^m$. Let $g = ab^{-1}$, so that $g^m = e$. This tells us that $|g| \mid m$. Moreover, by Lagrange's theorem, we know that $|g| \mid n$. Therefore, $|g|$ is a common divisor of m and n, implying $|g| = 1$ and hence $g = e$, as $\gcd(m, n) = 1$. In other words, $ab^{-1} = e$ and thus $a = b$, implying φ is injective. $\square$