Test 6 Math 301/601

Solutions

Instructions.

Problem 1. Let G be a group of order 4. Suppose there exist distinct order two elements $g, h \in G \setminus \{e\}$. Let $K = \langle g \rangle$ and $H = \langle h \rangle$. Prove that G is the internal direct product of H and K.

Solution. By inspection, $H \cap K = \{e\}$. We claim that $G = \{e, g, h, gh\}$. If $gh = e$, then $g = h$, but $g \neq h$ by assumptions, so $gh \neq e$. If $gh = g$, then $h = e$, but $h \neq e$, so $gh \neq g$. Similarly, $gh \neq h$. As $gh \notin \{e, g, h\}$ and as G has 4 elements, $G = \{e, g, h, gh\}$. We can therefore see that $G = KH$. Finally, a similar argument shows that $hg \notin \{e, g, h\}$, and hence $hg = gh$, as gh is the only remaining element in G. Therefore, every element of H commutes with each element of K. We can now conclude that G is the internal direct product of H and K. $\square$

Problem 2. Let $\varphi\colon G \to H$ be a homomorphism. Prove that $\ker\varphi$ is a normal subgroup of G.

Solution. Let $a \in \ker\varphi$ and let $g \in G$. Then

$$\begin{aligned}
\varphi(gag^{-1}) &= \varphi(g)\varphi(a)\varphi(g^{-1}) \\
&= \varphi(g)e_H\varphi(g^{-1}) \\
&= \varphi(g)\varphi(g^{-1}) \\
&= \varphi(gg^{-1}) \\
&= \varphi(e_G) \\
&= e_H
\end{aligned}$$

Therefore, $gag^{-1} \in \ker\varphi$ for all $a \in \ker\varphi$ and for all $g \in G$, implying that $\ker\varphi$ is normal. $\square$